2.3 THE WAVE EQUATION OF PLANE PROGRESSIVE (TRAVELLING) WAVE

The wave motion is due to the moving parts of a continuous system. (The system in which the mass is spread uniformly rather being concentrated at a finite number of points, is said to be continuous. An example, the vibrating string of a violin, where the mass is distributed along the length). The crests and troughs (in transverse waves) or compressions and rarefactions (in longitudinal waves) move forward while the moving parts of medium simply oscillate about their mean position and do not move. We shall consider plane wave which travel in one direction and establish a mathematical relation for the wave motion, called the wave equation.

In order to obtain such equation, consider a plane travelling harmonic wave originating at the origin O in a medium moving with a velocity v in the positive direction of x-axis, as shown in Fig. 2.6. As the wave proceeds, each successive particle of the medium is set into simple harmonic vibration of amplitude a. Let the time be measured from the instant when the particle at the origin O is passing through its equilibrium position. The displacement of a particle at O from its mean position at any time t is given by

$$\psi(0, t) = a \sin \omega t = a \sin \frac{2\pi}{T} t \qquad \because \omega = \frac{2\pi}{T} \qquad \text{...(2.1)}$$

where T is the periodic time and $\psi(0, t)$ is the displacement of particle at $x = 0$ at time t.

If now we consider a particle of the medium at a point A distant x from O, the wave starting from O would reach this point in (x/v) seconds. It means that this particle will start vibrating (x/v) seconds later than the particle at O. Therefore, there is a phase lag of x/v sec. between this particle and the particle at O. Consequently the displacement of the particle at A at a time t will be same as that of particle at O at a time x/v seconds earlier *i.e..*, at time $(t - x/v)$. Thus the displacement of a particle at A after a time t can be obtained by substituting $(t - x/v)$ in place of t in equation (2.1). i.e. displacement of the particle at A is

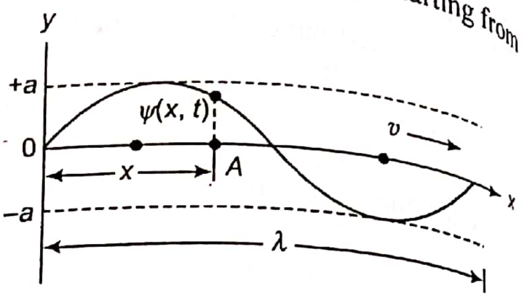

Fig. 2.6

$$\psi(x, t) = a \sin \frac{2\pi}{T} t^* \qquad \left(\text{where } t^* = t - \frac{x}{v}\right)$$

or

$$\psi(x, t) = a \sin \frac{2\pi}{T}\left(t - \frac{v}{x}\right) \qquad ...(2.2)$$

Equation (2.2) can also be written in the following forms:

$$\psi(x, t) = a \sin \frac{2\pi}{Tv} (v\,t - x)$$

$$\psi(x, t) = a \sin \frac{2\pi}{\lambda} (v\,t - x) \qquad [\because \lambda = T\,v] \qquad ...(2.3)$$

Again

$$\psi(x, t) = a \sin \frac{2\pi}{\lambda}\left(\frac{\lambda}{T} t - x\right)$$

or

$$\psi(x, t) = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad ...(2.4)$$

Equations (2.2), (2.3) and (2.4) represent the different forms of the displacement of a *plane progressive wave* of the continuous medium as a function of x and t.

For a wave travelling in the negative x-direction, the corresponding equation is

$$\psi(x, t) = a \sin \frac{2\pi}{T}\left(t + \frac{x}{v}\right) \qquad ...(2.5)$$

Equation (2.2) can be written in yet another compact form by defining two quantities, $k = 2\pi/\lambda$ and $\omega = 2\pi/T$. In terms of k and ω, we can write eqn. (2.2) as

$$\psi(x, t) = a \sin (\omega t - kx) \qquad ...(2.6)$$

where k is called the wave number and ω is the angular frequency of particle oscillations in a wave. Similarly the equation for a wave travelling in the negative x-direction can be expressed as

$$\psi(x, t) = a \sin (\omega t + kx) \qquad ...(2.7)$$

Properties of plane progressive wave. The equation of a plane progressive wave is given by

$$\psi(x, t) = a \sin \frac{2\pi}{\lambda} (v\,t - x) \qquad ...(2.8)$$

Ve note the following properties from this equation *(i)* All the particles vibrate in a simple harmonic manner with same amplitude of vibration a.

(ii) Keeping t constant, let the distance x be increased by x', then

$$\psi_1 = a \sin \frac{2\pi}{\lambda}\, [v\,t - (x + x')] = a \sin \left[\frac{2\pi}{\lambda}(vt - x) - \frac{2\pi}{\lambda} x'\right]$$

Further, if $x' = \lambda$, then

$$\psi_1 = a \sin \left[\frac{2\pi}{\lambda}(vt - x) - 2\pi\right]$$

$$= a \sin \frac{2\pi}{\lambda}\, (v\,t - x) = \psi(x, t)$$

Thus the displacement is the same when the distance is changed by λ. Hence λ is known as the wavelength of the wave.

(iii) Keeping x constant, let the time t be increased by t', then

$$\psi_2 = a \sin \frac{2\pi}{\lambda}\, [v\,(t + t') - x] = a \sin \left[\frac{2\pi}{\lambda}(vt - x) + \frac{2\pi}{\lambda} vt'\right]$$

Further, if $t' = \lambda/v$, then

$$\psi_2 = a \sin \left[\frac{2\pi}{\lambda}(vt - x) + 2\pi\right] = a \sin \frac{2\pi}{\lambda}\, (v\,t - x) = \psi$$

Thus the displacement of the particle after a time λ/v is again the same. Hence λ/v is known as the periodic time of the wave.

(iv) Let t be increased by δt and x is increased by $v\delta t$, then

$$\psi_2 = a \sin \frac{2\pi}{\lambda}\, v(t + \delta t) - (x + v\,\delta t)]$$

$$= a \sin \frac{2\pi}{\lambda}\, (v\,t - x) = \psi(x, t)$$

Thus in a time δt, the wave has advanced through a distance $v\ \delta t$. Hence v is the velocity of the wave.

Particle Velocity and Wave Velocity: The equation of displacement of a plane progressive wave is given by

$$\psi(x, t) = a \sin \frac{2\pi}{\lambda}\, (v\,t - x). \qquad \text{...(2.9)}$$

The particle velocity is defined as the rate of change of displacement y with respect to t, hence differentiating equation (2.9) with respect to t, we have

$$\frac{d\psi}{dt} = \frac{2\pi v a}{\lambda} \cos \frac{2\pi}{\lambda}\, (v\,t - x). \qquad \text{...(2.10)}$$

The maximum value of particle velocity is

$$\left(\frac{d\psi}{dt}\right)_{\max} = \frac{2\pi v a}{\lambda}, \qquad \text{when } \cos \frac{2\pi}{\lambda}\, (v\,t - x) = 1$$

$$\text{Maximum particle velocity} = \frac{2\pi a}{\lambda} \times (\text{wave velocity})$$

Differentiating equation (2.9) with respect to x, we have

$$\frac{d\psi}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}\, (v\,t - x). \qquad \text{...(2.11)}$$

$d\psi/dx$ represents the strain or compression. When $d\psi/dx$ is positive, a rarefaction takes place and when $d\psi/dx$ is negative, a compression takes place.

Comparing equations (2.10) and (2.11), we get

$$\text{Particle velocity} \quad \left(\frac{d\psi}{dt}\right) = \left(-v\frac{d\psi}{dx}\right) = \textit{wave velocity} \times \textit{slope of the displacement curve.} \qquad \text{...(2.12)}$$

This gives the *particle velocity* of the plane progressive wave.

Differentiating equation (2.11) again with respect to x, we have

$$\frac{d^2\psi}{dx^2} = -a\left(\frac{2\pi}{\lambda}\right)^2 \sin\frac{2\pi}{\lambda}(vt-x) \qquad \text{...(2.13)}$$

Again differentiating equation (2.10) with respect to t, we have

$$\frac{d^2\psi}{dt^2} = -a\left(\frac{2\pi}{\lambda}\right)^2 v^2 \sin\frac{2\pi}{\lambda}(vt-x) \qquad \text{...(2.14)}$$

This is the particle acceleration

Comparing equations (2.13) and (2.14), we get

$$\frac{d^2\psi}{dt^2} = v^2\frac{d^2\psi}{dx^2} \qquad \text{...(2.15)}$$

This is the one-dimensional *differential equation* of the wave motion and is called classical wave equation.

We may draw the following observations from wave equation (2.15):

(i) Wherever the second-order time derivative $d^2\psi/dt^2$ of any physical quantity is related to the second-order space derivative $d^2\psi/dx^2$, a wave of some sort travels in the medium.

(ii) The velocity v of that wave is given by the square–root of co-efficient of the second-order space derivative.

2.3.1 Phase and Phase Velocity

The individual particles which constitute the medium do not progress through the medium with the wave, they only oscillate about their mean positions. It is their phase relationship which we observe as waves. Thus the *wave velocity* is also called the *phase velocity*. It is the velocity with which plane waves of equal phase travel through the medium. The phase velocity of a wave is given by

$$v = \nu\lambda = 2\pi\nu\frac{\lambda}{2\pi}$$

or

$$v = \frac{\omega}{K} \qquad (\because \quad \omega = 2\pi v,\ k = 2\pi/\lambda) \qquad \text{...(2.16)}$$

The expression (2.16) can also be obtained in a different way. Consider the harmonic wave function (2.6) again

$$\psi(x, t) = a\sin(\omega t - kx)$$

which can also be written as

$$\psi(x, t) = a\sin(kx - \omega t) \qquad \text{...(2.17)}$$

The entire argument of the sine is the phase φ of the wave, where

$$\varphi = (kx - \omega t) \qquad \text{...(2.18)}$$

At $t = x = 0$, $\psi(x,t)\big|_{\substack{x=0\\t=0}} = \psi(0,0) = 0$

which is certainly a special case. More generally, we can write

$$\psi(x, t) = a \sin(kx - \omega t + \varepsilon) \qquad \ldots(2.19)$$

where ε is the *initial phase*. To get a sense of physical meaning of ε, imagine that we wish to produce a progressive harmonic wave on a stretched string, as in Fig. 2.7. In order to generate harmonic waves, the hand holding the string would have to move such that its vertical displacement y was proportional to the negative of its acceleration, that is, in simple harmonic motion. But at $t = 0$ and $x = 0$, the hand certainly need not be on the y-axis about to move downward, as in Fig. 2.7. It could, of course, begin its motion on an upward swing, in which case $\varepsilon = \pi$, as in Fig. 2.8. In this case,

$$\psi(x, t) = y(x, t) = a \sin(kx - \omega t + \pi)$$

which is equivalent to

$$\psi(x, t) = a \sin(\omega t - kx) \qquad \ldots(2.20)$$

or $\psi(x, t) = a \cos\left(\omega t - kx - \dfrac{\pi}{2}\right)$.

The initial phase angle is just the contribution to the phase arising at the generator and is independent of how far in space, or how long in time, the wave has traveled.

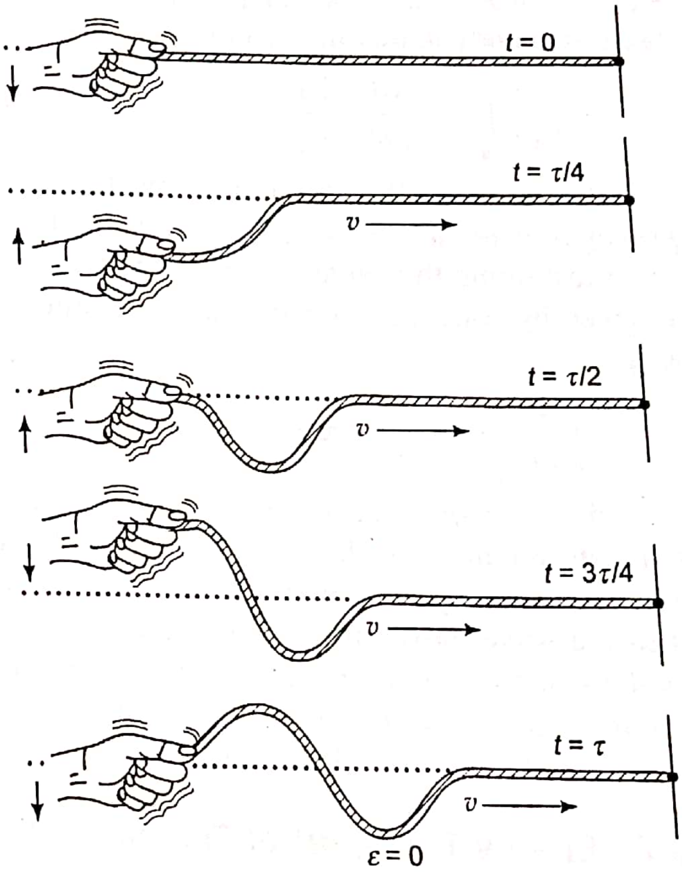

Fig. 2.7 *With* $\varepsilon = 0$ *note that at* $x = 0$ *and* $t = \tau/4 = \pi/2\omega$, $y = a\sin(-\pi/2) = -a$

The phase in eqn (2.17) is $(kx - \omega t)$, whereas in eqn (2.20) it is $(\omega t - kx)$. Nonetheless, both of these equation describe waves moving in the positive x-direction that are otherwise identical except for a relative phase difference of π. As is often the case, when the initial phase is of no particular significance in a given situation, either eqn. (2.17) or (2.20) or, if you like, a cosine function can be used to represent the wave.

The phase of a disturbance such as $\psi(x, t)$ given by eqn (2.19) is

$$\phi(x, t) = (kx - \omega t + \varepsilon)$$

and is obviously a function of x and t. In fact, the partial derivative of φ with respect to t, holding x constant, is the *rate-of-change of phase with time, or*

$$\left|\left(\frac{\partial \varphi}{\partial t}\right)_x\right| = \omega \qquad \ldots(2.21)$$

The rate-of-change of phase at any fixed location is the angular frequency of the wave, the rate at which a point on the rope in Fig. 2.7 oscillates up and down. That point must go through the same number of cycles per second as the wave. For each cycle, φ changes by 2π.

Similarly, the *rate-of-change of phase with distance*, holding t constant, is

$$\left|\left(\frac{\partial\varphi}{\partial x}\right)_t\right| = k \qquad \text{...(2.22)}$$

These two expressions should bring to mind an equation from the theory of partial derivatives, one used frequently in thermodynamics, namely,

$$\left(\frac{\partial x}{\partial t}\right)_\varphi = \frac{-(\partial\varphi/\partial t)_x}{(\partial\varphi/\partial x)_t} \qquad \text{...(2.23)}$$

The term on the left represents the *speed of propagation of the condition of constant phase.*

Substituting the values of partial derivatives of φ as given by eqns. (2.21) and (2.22) into eqn .(2.23), we get,

$$\left(\frac{\partial x}{\partial t}\right)_\varphi = \pm\frac{\omega}{k} = \pm v \qquad \text{...(2.24)}$$

This is the *speed* at which the profile moves and is known commonly as the *phase velocity* of the wave. The phase velocity is accompanied by a positive sign when the wave moves in the direction of increasing x and a negative one in the direction of decreasing x. This is consistent with our development of v as the magnitude of the wave velocity: $v > 0$.

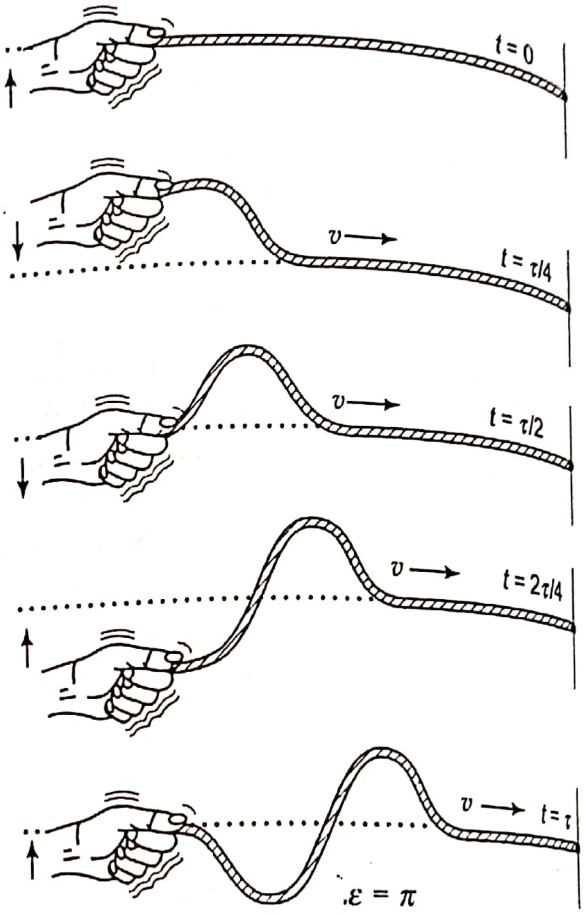

Fig. 2.8 *With $\varepsilon = \pi$ note that at $x = 0$ and $\tau = t/4$, $y = a \sin(\pi/2) = a$.*

2.3.2 Energy Transport of Travelling (Progressive) Wave

Kinetic energy: The equation of a plane progressive wave is given by

$$\psi = a \sin\frac{2\pi}{\lambda}(v\,t - x) \qquad \text{...(2.25)}$$

The velocity ($d\psi/dt$) of the particle at any time is

$$\frac{d\psi}{dt} = a\,\frac{2\pi}{\lambda}\,v \cos\frac{2\pi}{\lambda}(vt - x) \qquad \text{...(2.26)}$$

Consider a layer of unit area of cross section perpendicular to the direction of propagation of the wave. If ρ_0 be the density of the medium (mass per unit volume), then the mass of unit area of the layer of thickness dx will be $\rho_0\, dx$. The kinetic energy dK of this layer is given by

$$dK = \frac{1}{2} m \left(\frac{d\psi}{dt}\right)^2$$

$$= \frac{1}{2}\rho_0\, dx \,.\, a^2 \frac{4\pi^2}{\lambda^2} v^2 \cos^2\frac{2\pi}{\lambda}(v\,t - x) \qquad \text{...(2.27)}$$

The kinetic energy of the whole wave can be obtained by integrating eqn. (2.27) within the limits x and $(x + \lambda)$. Thus

$$K.E. = \int_x^{x+\lambda} \frac{1}{2}\rho_0 a^2 \frac{4\pi^2}{\lambda^2} v^2 \cos^2 \frac{2\pi}{\lambda}(vt - x)dx = \frac{2\pi^2 v^2 a^2 \rho_0}{\lambda^2}\int_x^{x+\lambda} \cos^2 \frac{2\pi}{\lambda}(vt-x)\,dx$$

$$= \frac{\omega^2 a^2 \rho_0}{4}\left(\int_x^{x+\lambda} 1 + \cos\frac{4\pi}{\lambda}(vt-x)dx\right) = \frac{\omega^2 a^2 \rho_0}{4}\left[\int_x^{x+\lambda} dx + \int_x^{x+\lambda}\cos\frac{4\pi}{\lambda}(vt-x)dx\right]$$

or

$$K.E. = \frac{\omega^2 a^2 \rho_0}{4}[\lambda + 0] = \frac{\omega^2 a^2 \rho_0 \lambda}{4} \qquad ...(2.28)$$

because

$$\int_x^{x+\lambda} \cos\frac{4\pi}{\lambda}(vt-x)dx = 0$$

Now kinetic energy per unit volume is given by

$$\text{K.E.} = \frac{1}{4}\omega^2 a^2 \rho_0 \qquad ...(2.29)$$

$\because$ Volume of the layer = unit area × wavelength = λ.

Potential energy: From eqn. (2.27), the kinetic energy per unit volume is given by

$$= \frac{1}{2}\rho_0 \frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2\frac{2\pi}{\lambda}(vt-x)$$

$$= \frac{1}{2}\rho_0 \omega^2 a^2 . \cos^2\frac{2\pi}{\lambda}(vt-x)$$

Now maximum kinetic energy per unit volume $= \frac{1}{2}\rho_0\omega^2 a^2 \quad \because \cos^2\frac{2\pi}{\lambda}(vt-x) = 1$

= Total energy of wave per unit volume ...(2.30)

($\because$ maximum kinetic energy = Total energy)

$\therefore$ Potential energy = Total energy – Kinetic energy

$$= \frac{1}{2}\rho_0\omega^2 a^2 - \frac{1}{2}\rho_0\omega^2 a^2\cos^2\frac{2\pi}{\lambda}(vt-x) \qquad ...(2.31)$$

By integrating eqn. (2.31) within the limits x and $(x+\lambda)$, the potential energy can be obtained.

$$\text{P.E.} = \frac{1}{2}\rho_0\omega^2 a^2 \int_x^{x+\lambda}\left[1 - \cos^2\frac{2\pi}{\lambda}(vt-x)\right]dx$$

$$= \frac{1}{2}\rho_0\omega^2 a^2 \int_x^{x+\lambda} \sin^2\frac{2\pi}{\lambda}(vt-x)\,dx$$

$$= \frac{1}{4}\rho_0\omega^2 a^2 \int_x^{x+\lambda}\left(1 - \cos\frac{4\pi}{\lambda}(vt-x)\right)dx$$

$$= \frac{1}{4}\rho_0\omega^2 a^2\lambda$$

Potential energy per unit volume $= \frac{1}{4}\rho_0\omega^2 a^2$

Total energy = K.E. + P.E.

$$\text{Total energy} = \frac{1}{4}\rho_0\omega^2 a^2 + \frac{1}{4}\rho_0\omega^2 a^2 = \frac{1}{2}\rho_0\omega^2 a^2 \quad \ldots(2.32)$$