Statistical mechanics

Lecture note: 3

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Topics in this Lectures:

1. White dwarf star
2. Mass Radius Relation.
3. Chandershekhar Mass limit.
4. Tutorial Problems.
5. Sci lab program.

White Dwarf Star:

- Very old star, almost in the end phase of their lives.
- much fainter, possess smaller diameter and are very dense.
- Example: Sirius B, 40 Eridani B.

Hertz–Russel Diagram: it's a plot of luminosity of a star versus temperature as shown in diagram. The white dwarf stars are located below the main sequence. Loss of brightness of white dwarf star is due to the fact that their hydrogen content has almost burnt out leading to formation of helium.

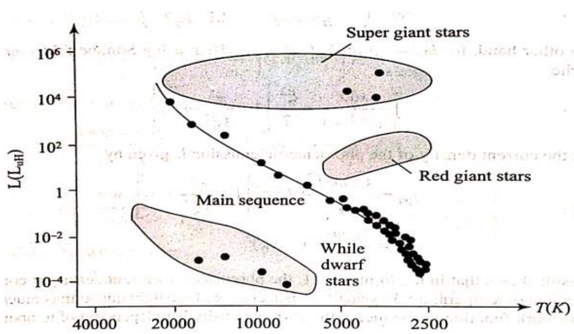

Figure 1 location of white dwarf star on HR diagram.

For example: white dwarf star generally has mass M = 10^{30} Kg.

Density = 10^{30} Kg/m^3 at T = 10^7 K.

This high temperature correspond to mean thermal energy of 10^3 eV, which is much higher than the energy required for ionizing Helium atom (=55 eV). In white dwarf star whole of helium is ionized and star is composed of He nuclei and electron. The gas of electrons behaves as an ideal Fermi gas. Electrons in typical dwarf star are relativistic.

Q. Which statistics we should follow, strongly degenerate or completely degenerate?

If there are N electrons then N/2 helium nuclei, so the mass of star:

$M = Nm_e + N/2 * 4 * m_p$

$M = N(m_e + 2m_p)$

$M = 2N\, m_p$

For sirius B, electron density = $1.7 * 10^{36}$ electrons/m^3.

So, Fermi energy: $\varepsilon_f = \frac{h^2}{8m}(\frac{3N}{\pi V})^{\frac{2}{3}}$ = 0.33 MeV

And Fermi temperature $T_f = \frac{\varepsilon_f}{k}$ = $3.8 * 10^9$ K

Temperature of Sirius B star = $2*10^7$ K.

Evaluating T/T_F = 0.005 which is >>1, thus electron gas is highly degenerate and we regard as ideal Fermi gas at T = 0. The dynamics of white dwarf is similar to the behavior of ideal Fermi Dirac gas of relativistic electrons.

Ground state energy and pressure of relativistic gas:

$\varepsilon_f = \frac{h^2}{2m_e}(\frac{3N}{8\pi V})^{\frac{2}{3}}$ .....(1)

$N = \frac{8\pi V}{3h^3}(\frac{m_e \varepsilon_F}{\pi V})^{\frac{3}{2}}$........(2)

Fermi momentum $P_F = (2m_e\varepsilon_F)^{1/2}$ ....(3)

From eqn. 1,2, and 3.

$$P_f = (\frac{3Nh^3}{8\pi V})^{\frac{1}{3}}$$

For relativistic particle $\varepsilon = \sqrt{P_l^2 c^2 + m_e^2 c^4}$

So ground state energy is given as

$$E_0 = \frac{2V}{h^3}\int_0^{P_F} \sqrt{P_l^2 c^2 + m_e^2 c^4} * 4\pi\, P_l^2\, dp_l$$

Solving the above equation we get,

$$E_0 = \frac{8\pi V m_e^4 c^5}{h^3} \int_0^{x_F} (1+x^2)^{1/2}\, x^2\, dx$$
$$= \frac{8\pi V m_e^4 c^5}{h^3} f(x_F)$$

Where $x_F = \frac{P_F}{m_{e*}c}$, and

$$f(x_F) = \int_0^{x_F} (1+x^2)^{1/2}\, x^2\, dx \begin{bmatrix} = \frac{1}{3}x_F^3\left(1+\frac{3}{10}x_F^2+\ldots\right) & \text{for } x_F \ll 1 \\ = \frac{1}{3}x_F^4\left(1+\frac{1}{x_F^2}+\ldots\right) & \text{for } x_F \gg 1 \end{bmatrix}$$

<u>Pressure of F-D gas:</u> The white dwarf star has huge pressure. For example Sirius B has 1.8*10^{22} N/m^2, which is very huge pressure. Such enormous pressure is counter balanced by gravitational attraction.

The pressure exerted by FD gas is

$$P_O = \frac{-dE_o}{dV}$$

Differentiating the ground state energy we get pressure

$$p_0 = \frac{8\pi m_e^4 c^5}{h^3}\begin{cases}\left[\frac{1}{3}x_F^3\left(1+\frac{1}{2}x_F^2+\ldots\right)-f(x_F)\right] \text{for } x_F \ll 1 \\ \left[\frac{1}{3}x_F^4\left(1+\frac{1}{2}x_F^{-2}+\ldots\right)-f(x_F)\right] \text{for } x_F \gg 1\end{cases}$$

$$p_0 = \begin{cases} \frac{8\pi m_e^4 c^5}{15h^3} x_F^5 & \text{for } x_F \ll 1 \\ \frac{2\pi m_e^4 c^5}{3h^3}(x_F^4 - x_F^2) & \text{for } x_F \gg 1 \end{cases}$$

<u>Mass Radius Relation</u>: mass and radius of white dwarf are inversely related.

Smaller the size of white dwarf, greater will be its mass. This means that in star, hydrogen is converted into helium and simultaneously star changes its size and become denser with passage of time.

For relativistic energy of electrons, the relation is given as:

$$\bar{R} = (\bar{M})^{1/3}\left[1-\left(\frac{\bar{M}}{M_0}\right)^{2/3}\right]^{1/2}$$

<u>Chandershekhar mass limit:</u> the above equation shows there exists limiting mass M_o which corresponds to very small size of white dwarf star. We conclude that no white star has mass greater than M_o.

Physical significance: if mass of white dwarf is greater than M_o than pressure of the system is insufficient and hence unable to support the gas against its tendency towards gravitational collapse. M_o is limiting mass of white dwarf. Chandershekhar relates limiting mass of white dwarf with mass of sun.

M_o = 1.44 * mass of sun

The above equation is known as Chandershekhar mass limit for white dwarf star.

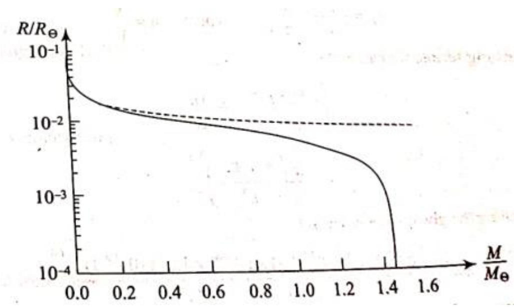

Figure 2 variation of radius of a white dwarf star with its mass. Both parameters are expressed in terms of solar radius and solar mass.

References:

1. *Thermal Physics by S.C. Garg, R.M. Bansal and C. K. Ghosh.*
2. *Statistical Physics, Berkley series volume 5.*
3. *Fundaments of Statitical Mechanics and Thermal Physics, F. Rief.*

Tutorial problems:

Q1. Prove that for a system at a finite temperature and obeying FD statistics, the probability that a level lying ΔE below the Fermi level is unoccupied is same as the probability of occupation of a level lying ΔE below the Fermi level.

Q2. Calculate the Fermi wavelength and Fermi energy for 4.2 *10^{21} electrons in a box of volume 1 cm^3. If the electrons are replaced by neutrons, how do Fermi wavelength and Fermi energy change.

SCI LAB Programs:

Program 1: Plot density of states, occupation number and distribution function of particles in non-relativistic Fermi gas and Bose gas for

(a) Low temperature = 20 K

(b) High temperature = 800 K.

Program 2: Plot the temperature variation of chemical potential of Fermi gas.

$$\mu(T) = \varepsilon_f \left[1 + \frac{\pi^2}{8}\left(\frac{T}{T_f}\right)^2 + \cdots\right]^{\frac{2}{3}}$$

on x axis: T

on Y axis: μ

take $\varepsilon_f = 1, T_f = 1.$