Introduction to Electrodynamics

David J. Griffiths
Reed College

Prentice Hall
Upper Saddle River, New Jersey 07458

5.3 The Divergence and Curl of B

5.3.1 Straight-Line Currents

The magnetic field of an infinite straight wire is shown in Fig. 5.27 (the current is coming *out* of the page). At a glance, it is clear that this field has a nonzero curl (something you'll *never* see in an *electro*static field); let's *calculate* it.

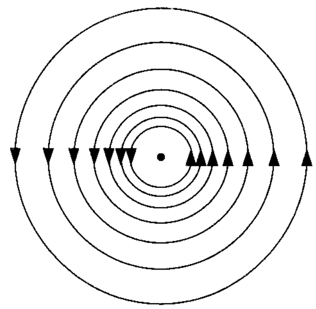

Figure 5.27

According to Eq. 5.36, the integral of **B** around a circular path of radius s, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

Notice that the answer is independent of s; that's because B *de*creases at the same rate as the circumference *in*creases. In fact, it doesn't have to be a circle; *any* old loop that encloses the wire would give the same answer. For if we use cylindrical coordinates (s, ϕ, z), with the current flowing along the z axis,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}, \tag{5.41}$$

and $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$, so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s}\, s\, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

This assumes the loop encircles the wire exactly once; if it went around twice, the ϕ would run from 0 to 4π, and if it didn't enclose the wire at all, then ϕ would go from ϕ_1 to ϕ_2 and back again, with $\int d\phi = 0$ (Fig. 5.28).

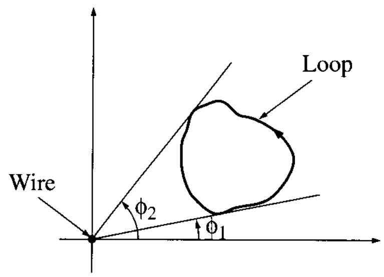

Figure 5.28

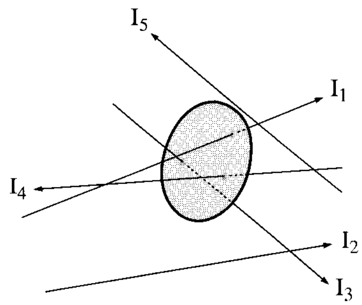

Figure 5.29

Now suppose we have a *bundle* of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing (Fig. 5.29). The line integral will then be

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}, \tag{5.42}$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density $\mathbf{J}$, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}, \tag{5.43}$$

with the integral taken over the surface bounded by the loop. Applying Stokes' theorem to Eq. 5.42, then,

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{5.44}$$

With minimal labor we have actually obtained the general formula for the curl of $\mathbf{B}$. But our derivation is seriously flawed by the restriction to infinite straight line currents (and combinations thereof). Most current configurations *cannot* be constructed out of infinite straight wires, and we have no right to assume that Eq. 5.44 applies to them. So the next section is devoted to the formal derivation of the divergence and curl of $\mathbf{B}$, starting from the Biot-Savart law itself.

5.3.2 The Divergence and Curl of B

The Biot-Savart law for the general case of a volume current reads

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} \, d\tau'. \tag{5.45}$$

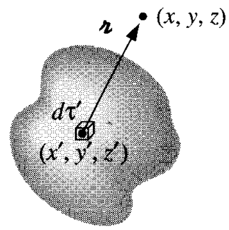

Figure 5.30

This formula gives the magnetic field at a point $\mathbf{r} = (x, y, z)$ in terms of an integral over the current distribution $\mathbf{J}(x', y', z')$ (Fig. 5.30). It is best to be absolutely explicit at this stage:

$$\mathbf{B} \text{ is a function of } (x, y, z),$$

$$\mathbf{J} \text{ is a function of } (x', y', z'),$$

$$\mathscr{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the *primed* coordinates; the divergence and the curl are to be taken with respect to the *unprimed* coordinates.

Applying the divergence to Eq. 5.45, we obtain:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.46}$$

Invoking product rule number (6),

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right). \tag{5.47}$$

But $\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables (x, y, z), whereas $\nabla \times (\hat{\mathscr{r}}/\mathscr{r}^2) = 0$ (Prob. 1.62), so

$$\boxed{\nabla \cdot \mathbf{B} = 0.} \tag{5.48}$$

Evidently, the *divergence* of the magnetic field is *zero*.

Applying the curl to Eq. 5.45, we obtain:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'. \tag{5.49}$$

Again, our strategy is to expand the integrand, using the appropriate product rule—in this case number 8:

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}. \tag{5.50}$$

(I have dropped terms involving derivatives of $\mathbf{J}$, because $\mathbf{J}$ does not depend on x, y, z.) The second term integrates to zero, as we'll see in the next paragraph. The first term involves the divergence we were at pains to calculate in Chapter 1 (Eq. 1.100):

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}). \tag{5.51}$$

Thus

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau' = \mu_0\mathbf{J}(\mathbf{r}),$$

which confirms that Eq. 5.44 is not restricted to straight-line currents, but holds quite generally in magnetostatics.

To complete the argument, however, we must check that the second term in Eq. 5.50 integrates to zero. Because the derivative acts only on $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, we can switch from ∇ to ∇' at the cost of a minus sign:[9]

$$-(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J}\cdot\nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.52}$$

The x component, in particular, is

$$(\mathbf{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)(\nabla'\cdot\mathbf{J})$$

(using product rule 5). Now, for *steady* currents the divergence of $\mathbf{J}$ is zero (Eq. 5.31), so

$$\left[-(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]_x = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right],$$

and therefore this contribution to the integral (5.49) can be written

$$\int_{\mathcal{V}} \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau' = \oint_{\mathcal{S}} \frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\cdot d\mathbf{a}'. \tag{5.53}$$

(The reason for switching from ∇ to ∇' was precisely to permit this integration by parts.) But what region are we integrating over? Well, it's the volume that appears in the Biot-Savart law (5.45)—large enough, that is, to include all the current. You can make it *bigger* than that, if you like; $\mathbf{J} = 0$ out there anyway, so it will add nothing to the integral. The essential point is that *on the boundary* the current is *zero* (all current is safely *inside*) and hence the surface integral (5.53) vanishes.[10]

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[9] The point here is that $\boldsymbol{\mathscr{r}}$ depends only on the *difference* between the coordinates, and $(\partial/\partial x)f(x-x') = -(\partial/\partial x')f(x-x')$.

[10] If $\mathbf{J}$ itself extends to infinity (as in the case of an infinite straight wire), the surface integral is still typically zero, though the analysis calls for greater care.

5.3.3 Applications of Ampère's Law

The equation for the curl of **B**

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},} \tag{5.54}$$

is called **Ampère's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

Now, $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface (Fig. 5.31), which we call I_{enc} (the **current enclosed** by the **amperian loop**). Thus

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}.} \tag{5.55}$$

This is the integral version of Ampère's law; it generalizes Eq. 5.42 to *arbitrary* steady currents. Notice that Eq. 5.55 inherits the sign ambiguity of Stokes' theorem (Sect. 1.3.5): Which *way* around the loop am I supposed to go? And which *direction* through the surface corresponds to a "positive" current? The resolution, as always, is the right-hand rule: If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.

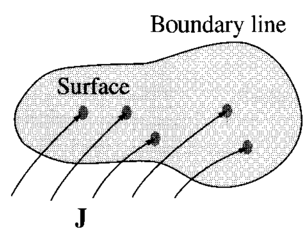


Figure 5.31

Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, so Ampère's plays the role of Gauss's:

$$\begin{cases} \text{Electrostatics}: & \text{Coulomb} & \rightarrow & \text{Gauss}, \\ \text{Magnetostatics}: & \text{Biot-Savart} & \rightarrow & \text{Ampère}. \end{cases}$$

In particular, for currents with appropriate symmetry, Ampère's law in integral form offers a lovely and extraordinarily efficient means for calculating the magnetic field.

Example 5.7

Find the magnetic field a distance s from a long straight wire (Fig. 5.32), carrying a steady current I (the same problem we solved in Ex. 5.5, using the Biot-Savart law).

Solution: We know the direction of **B** is "circumferential," circling around the wire as indicated by the right hand rule. By symmetry, the magnitude of **B** is constant around an amperian loop of radius s, centered on the wire. So Ampère's law gives

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\oint dl = B2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

or

$$B = \frac{\mu_0 I}{2\pi s}.$$

This is the same answer we got before (Eq. 5.36), but it was obtained this time with far less effort.

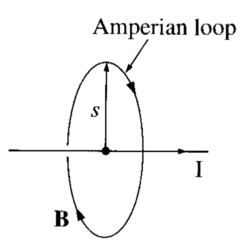


Figure 5.32

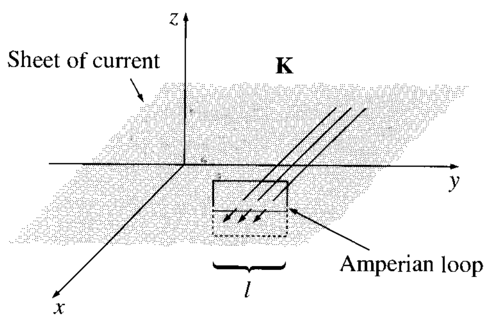


Figure 5.33

Example 5.8

Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).

Solution: First of all, what is the *direction* of **B**? Could it have any x-component? *No*: A glance at the Biot-Savart law (5.39) reveals that **B** is *perpendicular* to **K**. Could it have a z-component? *No again.* You could confirm this by noting that any vertical contribution from a filament at $+y$ is canceled by the corresponding filament at $-y$. But there is a nicer argument: Suppose the field pointed *away* from the plane. By reversing the direction of the current, I could make it point *toward* the plane (in the Biot-Savart law, changing the sign of the current switches the sign of the field). But the z-component of **B** cannot possibly depend on the *direction* of the current in the xy plane. (Think about it!) So **B** can only have a y-component, and a quick check with your right hand should convince you that it points to the *left* above the plane and to the *right* below it.

With this in mind we draw a rectangular amperian loop as shown in Fig. 5.33, parallel to the yz plane and extending an equal distance above and below the surface. Applying Ampère's law, we find

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

(one Bl comes from the top segment, and the other from the bottom), so $B = (\mu_0/2)K$, or, more precisely,

$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases} \tag{5.56}$$

Notice that the field is independent of the distance from the plane, just like the *electric* field of a uniform surface *charge* (Ex. 2.4).

Example 5.9

Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I (Fig. 5.34). [The point of making the windings so close is that one can then pretend each turn is circular. If this troubles you (after all, there is a net current I in the direction of the solenoid's axis, no matter *how* tight the winding), picture instead a sheet of aluminum foil wrapped around the cylinder, carrying the equivalent uniform surface current $K = nI$ (Fig. 5.35). Or make a double winding, going up to one end and then—always in the same sense—going back down again, thereby eliminating the net longitudinal current. But, in truth, this is all unnecessary fastidiousness, for the field inside a solenoid is huge (relatively speaking), and the field of the longitudinal current is at most a tiny refinement.]

Solution: First of all, what is the *direction* of **B**? Could it have a radial component? *No.* For suppose B_s were *positive*; if we reversed the direction of the current, B_s would then be *negative*. But switching I is physically equivalent to turning the solenoid upside down, and

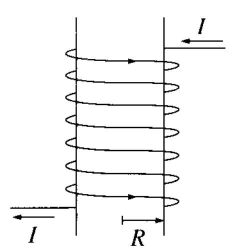


Figure 5.34

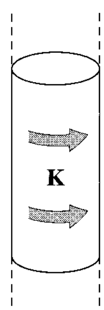


Figure 5.35

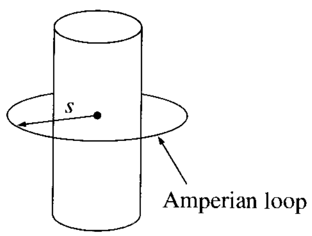


Figure 5.36

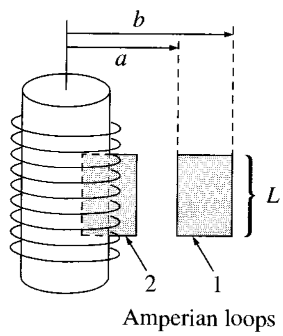


Figure 5.37

that certainly should not alter the radial field. How about a "circumferential" component? *No.* For B_ϕ would be constant around an amperian loop concentric with the solenoid (Fig. 5.36), and hence

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi(2\pi s) = \mu_0 I_{\text{enc}} = 0,$$

since the loop encloses no current.

So the magnetic field of an infinite, closely wound solenoid runs *parallel to the axis.* From the right hand rule, we expect that it points upward inside the solenoid and downward outside. Moreover, it certainly approaches zero as you go very far away. With this in mind, let's apply Ampère's law to the two rectangular loops in Fig. 5.37. Loop 1 lies entirely outside the solenoid, with its sides at distances a and b from the axis:

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0,$$

so

$$B(a) = B(b).$$

Evidently the *field outside does not depend on the distance from the axis.* But we know that it goes to *zero* for large s. It must therefore be zero *everywhere*! (This astonishing result can also be derived from the Biot-Savart law, of course, but it's much more difficult. See Prob. 5.44.)

As for loop 2, which is half inside and half outside, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L,$$

where B is the field inside the solenoid. (The right side of the loop contributes nothing, since $B = 0$ out there.) *Conclusion:*

$$\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid,} \\ 0, & \text{outside the solenoid.} \end{cases} \tag{5.57}$$

Notice that the field inside is *uniform*; in this sense the solenoid is to magnetostatics what the parallel-plate capacitor is to electrostatics: a simple device for producing strong uniform fields.

Like Gauss's law, Ampère's law is always *true* (for steady currents), but it is not always *useful*. Only when the symmetry of the problem enables you to pull B outside the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ can you calculate the magnetic field from Ampère's law. When it *does* work, it's by far the fastest method; when it doesn't, you have to fall back on the Biot-Savart law. The current configurations that can be handled by Ampère's law are

1. Infinite straight lines (prototype: Ex. 5.7).
2. Infinite planes (prototype: Ex. 5.8).
3. Infinite solenoids (prototype: Ex. 5.9).
4. Toroids (prototype: Ex. 5.10).

The last of these is a surprising and elegant application of Ampère's law; it is treated in the following example. As in Exs. 5.8 and 5.9, the hard part is figuring out the *direction* of the field (which we will now have done, once and for all, for each of the four geometries); the actual application of Ampère's law takes only one line.

Example 5.10

A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a closed loop. The cross-sectional shape of the coil is immaterial. I made it rectangular in Fig. 5.38 for the sake of simplicity, but it could just as well be circular or even some weird asymmetrical form, as in Fig. 5.39, just as long as the shape remains the same all the way around the ring. In that case it follows that the *magnetic field of the toroid is circumferential at all points, both inside and outside the coil.*

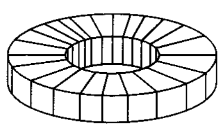


Figure 5.38

Proof: According to the Biot-Savart law, the field at $\mathbf{r}$ due to the current element at $\mathbf{r}'$ is

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{\mathbf{I} \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} dl'.$$

We may as well put $\mathbf{r}$ in the xz plane (Fig. 5.39), so its Cartesian components are $(x, 0, z)$, while the source coordinates are

$$\mathbf{r}' = (s' \cos\phi', s' \sin\phi', z').$$

Then

$$\boldsymbol{\mathscr{r}} = (x - s' \cos\phi', -s' \sin\phi', z - z').$$

Since the current has no ϕ component, $\mathbf{I} = I_s\,\hat{\mathbf{s}} + I_z\,\hat{\mathbf{z}}$, or (in Cartesian coordinates)

$$\mathbf{I} = (I_s \cos\phi', I_s \sin\phi', I_z).$$

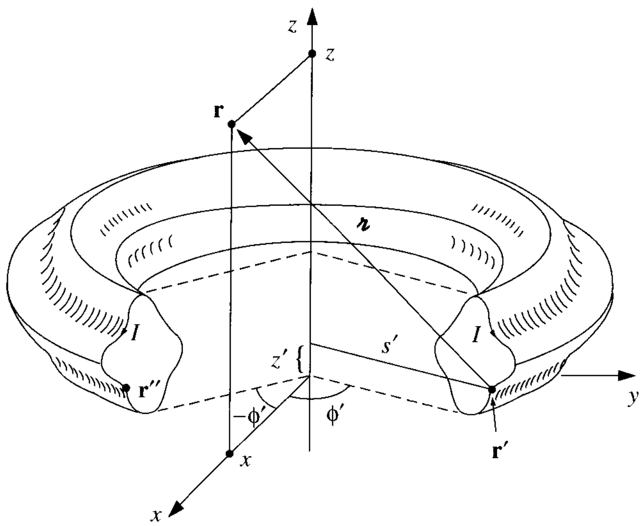


Figure 5.39

Accordingly,

$$\mathbf{I}\times\boldsymbol{\mathscr{r}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s\cos\phi' & I_s\sin\phi' & I_z \\ (x-s'\cos\phi') & (-s'\sin\phi') & (z-z') \end{bmatrix}$$

$$= [\sin\phi'(I_s(z-z')+s'I_z)]\,\hat{\mathbf{x}}$$

$$+ [I_z(x-s'\cos\phi') - I_s\cos\phi'(z-z')]\,\hat{\mathbf{y}} + [-I_s x\sin\phi']\,\hat{\mathbf{z}}.$$

But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', the same $\mathscr{r}$, the same dl', the same I_s, and the same I_z, *but negative* ϕ' (Fig. 5.39). Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term. Thus the field at $\mathbf{r}$ is in the $\hat{\mathbf{y}}$ direction, and in general the field points in the $\hat{\boldsymbol{\phi}}$ direction. qed

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B 2\pi s = \mu_0 I_{\text{enc}},$$

and hence

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ 0, & \text{for points outside the coil,} \end{cases} \tag{5.58}$$

where N is the total number of turns.

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.

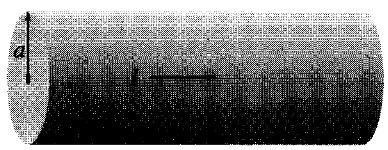


Figure 5.40

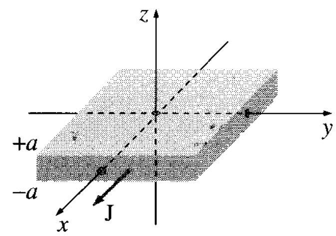


Figure 5.41

Problem 5.14 A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J\,\hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z, both inside and outside the slab.

Problem 5.15 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\mathbf{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

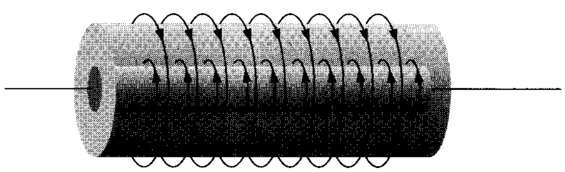


Figure 5.42

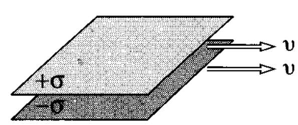


Figure 5.43

Problem 5.16 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[11]

[11] See footnote 8.

! **Problem 5.17** Show that the magnetic field of an infinite solenoid runs parallel to the axis, *regardless of the cross-sectional shape of the coil*, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? Show that the toroid field (5.58) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.

Problem 5.18 In calculating the current enclosed by an amperian loop, one must, in general, evaluate an integral of the form

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{a}.$$

The trouble is, there are infinitely many surfaces that share the same boundary line. Which one are we supposed to use?

5.3.4 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the *electrostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$

These are **Maxwell's equations** for electrostatics. Together with the boundary condition $\mathbf{E} \to 0$ far from all charges, Maxwell's equations determine the field, if the source charge density ρ is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the *magnetostatic* field are

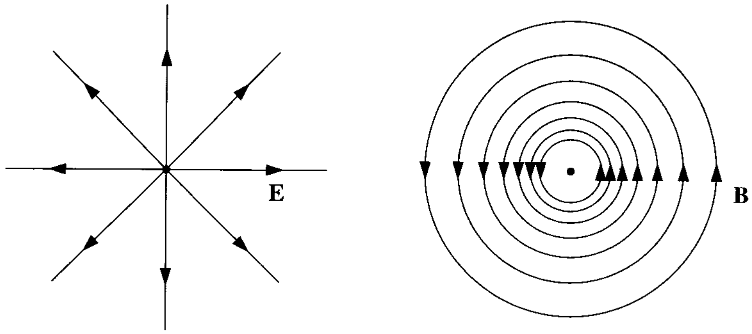
$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to 0$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition). Maxwell's equations and the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current (Fig. 5.44). Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere—to do so would require a nonzero divergence. They either form closed loops or extend out to infinity. To put it another way, *there are no point sources for* **B**, as there are for **E**; there exists no magnetic analog to electric charge. This is the physical content of the statement $\nabla \cdot \mathbf{B} = 0$. Coulomb and others believed that magnetism was produced by **magnetic charges** (**magnetic monopoles**, as we would now call them), and in some older books you will still



(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

Figure 5.44

find references to a magnetic version of Coulomb's law, giving the force of attraction or repulsion between them. It was Ampère who first speculated that all magnetic effects are attributable to *electric* charges *in motion* (currents). As far as we know, Ampère was right; nevertheless, it remains an open experimental question whether magnetic monopoles exist in nature (they are obviously pretty *rare*, or somebody would have found one[12]), and in fact some recent elementary particle theories *require* them. For our purposes, though, **B** is divergenceless and there are no magnetic monopoles. It takes a *moving* electric charge to *produce* a magnetic field, and it takes another moving electric charge to "feel" a magnetic field.

Typically, electric forces are enormously larger than magnetic ones. That's not something you can tell from the theory as such; it has to do with the sizes of the fundamental constants ϵ_0 and μ_0. In general, it is only when both the source charges and the test charge are moving at velocities comparable to the speed of light that the magnetic force approaches the electric force in strength. (Problems 5.12 and 5.16 illustrate this rule.) How is it, then, that we ever notice magnetic effects at all? The answer is that both in the production of a magnetic field (Biot-Savart) and in its detection (Lorentz) it is the *current* (charge times velocity) that enters, and we can compensate for a smallish velocity by pouring huge quantities of charge down the wire. Ordinarily, this charge would simultaneously generate so large an *electric* force as to swamp the magnetic one. But if we arrange to keep the wire *neutral*, by embedding in it an equal amount of opposite charge at rest, the electric field cancels out, leaving the magnetic field to stand alone. It sounds very elaborate, but of course this is precisely what happens in an ordinary current carrying wire.

[12]An apparent detection (B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982)) has never been reproduced—and not for want of trying. For a delightful brief history of ideas about magnetism, see Chapter 1 in D. C. Mattis, *The Theory of Magnetism* (New York: Harper and Row, 1965).