Summary of Numerical Integration: (quadrature Rule)

There are two types of quadrature Rule or Numerical Integration.

(I) closed quadrature Rule:

(II) open quadrature Rule:

In this method do not include the endpoints of the integration interval among the abscissas.

for these formula,

$$\Delta x = \left(\frac{b-a}{n+2}\right) \text{ and}$$

$$x_i = a + (i+1)\Delta x, \; i = 0, 1, 2, \ldots n$$

In this method include the endpoints of the integration interval, $x = a$ and $x = b$, among the abscissas.

For given n, we take

$$\Delta x = (b-a)/n$$

$$x_i = a + i\Delta x, \; i = 0, 1, 2, \ldots n.$$

Trapezoidal Rule: for $n=1$.

$$I(f) \simeq I_{1,\,closed}(f) = \frac{\Delta x}{2}\left[f(a) + f(b)\right]$$

$$\boxed{I_{1,\,closed}(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$

Simpson's Rule: for $n=2$:

$$I(f) \simeq I_{2,\,closed}(f) = \frac{\Delta x}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\boxed{I_{2,\,closed}(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Midpoint Rule : $n=0$

$$\boxed{I(f) \approx I_{0,opened}(f) = (b-a)\, f\left(\frac{a+b}{2}\right)}$$

for $n=1$:

$$\boxed{I_{1,open}(f) = \frac{b-a}{2}\left[f(a+\Delta x) + f(a+2\Delta x)\right]}$$

for $n=2$:

$$I_{2,open}(f) = \frac{b-a}{2}\left[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)\right]$$

Where $\Delta x = \frac{b-a}{4}$

Example: find the approximate value of $\int_1^2 \frac{1}{x}\,dx$ by Simpson's Rule and Trapezoidal Rule.

Solⁿ: here $a=1$, $b=2$, $f(x)=\frac{1}{x}$

by Trapezoidal Rule

$$I_{1,closed} = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

$$= \frac{2-1}{2}\left[f(1)-f(2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{1}-\frac{1}{2}\right]$$

$$= \frac{1}{2}\times 0.5 = \underline{0.25}$$

by Simpson's Rule

$$I_{2,closed} = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$I_{2,closed}(f) = \frac{2-1}{6}\left[f(1) + 4 f\left(\tfrac{3}{2}\right) + f(2)\right]$$

$$= \frac{1}{6}\left[1 + 4\cdot\left(\tfrac{2}{3}\right) + \tfrac{1}{2}\right]$$

$$= \frac{1}{6}\left[1 + \frac{8}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{6}\left[\frac{6+16+3}{6}\right]$$

$$= \frac{1}{6}\times\frac{25}{6} = \frac{25}{36} = \underline{\underline{0.6944}}$$

useful for certain problems with endpoint singularities. Open formulas also find use in the construction of numerical methods for the solution of initial value problems.

EXERCISES

Q.1. Approximate the value of each of the following integrals using the trapezoidal rule. Verify that the theoretical error bound holds in each case.

(a) $\int_1^2 \frac{1}{x}\, dx$

(b) $\int_0^1 e^{-x}\, dx$

(c) $\int_0^1 \frac{1}{1+x^2}\, dx$

(d) $\int_0^1 \tan^{-1} x\, dx.$

Q.2. Repeat Exercise 1 using Simpson's rule rather than the trapezoidal rule.

3. Repeat Exercise 1 using the midpoint rule rather than the trapezoidal rule.

4. Verify directly that the midpoint rule has degree of precision equal to 1.

5. Verify directly that the open Newton-Cotes formula with $n = 1$ has degree of precision equal to 1.

6. (a) Determine values for the coefficients A_0, A_1, and A_2 so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)\, dx = A_0 f\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$$

has degree of precision at least 2.

(b) Once the values of A_0, A_1, and A_2 have been computed, determine the overall degree of precision for the quadrature rule.

7. (a) Determine values for the coefficients A_0, A_1, and A_2 so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)\, dx = A_0 f\left(-\frac{1}{3}\right) + A_1 f\left(\frac{1}{3}\right) + A_2 f(1)$$

has degree of precision at least 2.

(b) Once the values of A_0, A_1, and A_2 have been computed, determine the overall degree of precision for the quadrature rule.

8. (a) Determine values for the coefficients A_0, A_1, and x_1 so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)\, dx = A_0 f(-1) + A_1 f(x_1)$$

has degree of precision at least 2.

(b) Once the values of A_0, A_1, and x_1 have been computed, determine the overall degree of precision for the quadrature rule.

9. Consider the quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

Determine the degree of precision of this formula.

10. Consider the quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Determine the degree of precision of this formula.

11. Derive the error term for the midpoint rule:

$$\frac{(b-a)^3}{24} f''(\xi),$$

where $a < \xi < b$.

12. (a) Derive the closed Newton-Cotes formula with $n = 3$:

$$I(f) \approx I_{3,\text{closed}}(f) = \frac{b-a}{8}[f(a) + 3f(a+\Delta x) + 3f(a+2\Delta x) + f(b)].$$

(b) Verify that this formula has degree of precision equal to 3.

(c) Derive the error term associated with this quadrature rule.

13. (a) Derive the closed Newton-Cotes formula with $n = 4$:

$$I(f) \approx I_{4,\text{closed}}(f) = \frac{b-a}{90}[7f(a) + 32f(a+\Delta x) + 12f(a+2\Delta x) + 32f(a+3\Delta x) + 7f(b)]$$

(b) Verify that this formula has degree of precision equal to 5.

(c) Derive the error term associated with this quadrature rule.

14. (a) Derive the open Newton-Cotes formula with $n = 2$:

$$I(f) \approx I_{2,\text{open}}(f) = \frac{b-a}{3}[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)].$$

(b) Verify that this formula has degree of precision equal to 3.

(c) Derive the error term associated with this quadrature rule.

15. (a) Derive the open Newton-Cotes formula with $n = 3$:

$$I(f) \approx I_{3,\text{open}}(f) = \frac{b-a}{24}[11f(a+\Delta x) + f(a+2\Delta x) + f(a+3\Delta x) + 11f(a+4\Delta x)].$$

(b) Verify that this formula has degree of precision equal to 3.

Course – GE-4
Paper Name – Numerical methods.
Name of Teacher: Virendra Kumar.

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Piecewise Interpolation: (Piecewise linear Interpolation)
Let "f" be c/s f^n defined on closed interval $[a,b]$. The interval $[a,b]$ The interval $[a,b]$ divided into subintervals divided into $(n+1)$ distinct points. The interval $[a,b]$ divided into subintervals
$[x_{i-1}, x_i]$ containing 2 nodal points.
We have $x \in [x_{i-1}, x_i]$, the piecewise linear interpolation.

$$P_{i,1}(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i), \quad i = 1, 2, \ldots n$$

For $x \in [x_i, x_{i+1}]$, we have

$$P_{i+1,1}(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1})$$

Define

$$N_i(x) = \begin{cases} 0, & x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1} \\ 0, & x \ge x_{i+1} \end{cases}$$

Note that the non-zero terms in $N_i(x)$ are the coefficients of $f(x_i)$ in $P_{i,1}(x)$ and $P_{i+1,1}(x)$ respectively.
Then the interpolating polynomial

$$p(x) = \sum_{i=0}^{n} P_{i,1}(x)$$

or

$$\boxed{P_1(x) = \sum_{i=0}^{n} N_i(x) f(x_i)}$$

Piecewise. Linear int. poly

Example: Obtain the piecewise linear interpolating polynomials for the function f(x) defined by the data.

x	1	2	4	8
$f(x)$	3	7	21	73

Hence, estimate the values of $f(3)$ and $f(7)$.

Sol^n: Given the node points divided into subintervals.
let $I_1 = [1,2]$, $I_2 = [2,4]$, $I_3 = [4,8]$.

In the interval $[1,2]$, we have

$$P_1(x) = \frac{x-2}{1-2} f(1) + \frac{x-1}{2-1} f(2) \quad \text{(by def.)}$$

$$= \frac{x-2}{(-1)} 3 + \frac{x-1}{1} \times 7$$

$$\boxed{P_1(x) = 4x-1}$$

In the interval $[2,4]$, we have

$$P_1(x) = \frac{x-4}{-2}(7) + \frac{(x-2)}{2}(21) = 7x-7$$

$$\boxed{P_1(x) = 7x-7}$$

In the interval $[4,8]$, we have

$$P_1(x) = \frac{x-8}{-4}(21) + \frac{x-4}{4}(73).$$

$$\boxed{P_1(x) = 13x-31}$$

Hence, the piecewise linear interpolating poly. is given by

$$P_1(x) = \begin{cases} 4x-1, & 1 \le x \le 2 \\ 7x-7, & 2 \le x \le 4 \\ 13x-31, & 4 \le x \le 8 \end{cases}$$

Using the polynomial in the interval [2, 4],

we obtain $f(3) = 21 - 7 = 14$.

Using the polynomial in the interval [4, 8],

we obtain. $f(7) = 91 - 31 = 60$.

Ans.

~~Spline Interpolation~~:

Problems:

(1) obtain the piecewise ~~quadratic~~ linear interpolating polynomials for the f(x) defined by the data.

x	-3	-2	-1	1	3	6	7
$f(x)$	369	222	171	165	207	990	1779

Hence, find approximate value of $f(-2.5)$ and $f(6.5)$.

Ans. $f(-2.5) = 283.5$, $f(6.5) \approx 1351.5$.

Q.2. (Exam problem).

obtain the piecewise linear interpolating polynomial for the function f(x) defined by the given data

x	0	1	2	3
$f(x)$	1	2	5	10

and interpolate at $x = 0.5$ and 1.5.

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GE-4

Cubic Spline Interpolation:

A cubic spline satisfies the following properties:

(i) $F(x_i) = f_i$, $i = 0, 1, 2, \ldots n$.

(ii) on each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, $F(x)$ is a third degree polynomial.

(iii) $F(x)$, $F'(x)$ and $F''(x)$ are continuous on (a, b).

we denote $F'(x_i) = m_i$ and $F''(x_i) = M_i$

The cubic spline interpolation is

$$F(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(f_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6}M_i\right)$$

and $$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(f_{i+1} - 2f_i + f_{i-1}\right).$$

Note : This method is for data is equally spaced.

Example : obtain the cubic spline approximation for the f^n defined by the data

x	0	1	2	3
$f(x)$	1	2	33	244

with $M(0) = 0$, $M(3) = 0$, Hence find an estimate of $f(2.5)$

Solⁿ: We have data points

x	0	1	2	3
$f(x)$	1	2	33	244

system are equally spaced i.e

$x_i = x_0 + ih$, $i = 0, 1, 2, \ldots$

$\exists$ $h = 1$ for $i = 1$.

We have, $I_1 = [0, 1]$, $I_2 = [1, 2]$, $I_3 = [2, 3]$

by cubic method:

$M_{i-1} + 4M_i + M_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1})$, $i = 1, 2$.

for $i = 1$,

$$M_0 + 4M_1 + M_2 = 6(f_2 - 2f_1 + f_0)$$

for $i = 2$, $M_1 + 4M_2 + M_3 = 6(f_3 - 2f_2 + f_1)$

Using $M(0) = M_0 = 0$, $M(3) = M_3 = 0$ and the given function values, we get

$$4M_1 + M_2 = 6(33 - 4 + 1) = 180$$

$$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$$

solving for M_1 and M_2,

$$M_1 = -24, \quad M_2 = 276$$

Now, by cubic spline method,

$$F(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(f_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6}M_i\right)$$

on $[0,1]$:

$$F(x) = \frac{1}{6}\left[(1-x)^3 M_0 + (x-0)^3 M_1\right] + (1-x)\left(f_0 - \frac{1}{6}M_0\right) + (x-0)\left(f_1 - \frac{1}{6}M_1\right)$$

$$= \frac{1}{6}x^3(-24) + (1-x) + x\left[2 - \frac{1}{6}(-24)\right)$$

$$= -4x^3 + 5x + 1$$

on $[1,2]$: $F(x) = \frac{1}{6}\left[(2-x)^3 M_1 + (x-1)^3 M_2\right] + (2-x)\left(f_1 - \frac{1}{6}M_1\right) + (x-1)\left(f_2 - \frac{1}{6}M_2\right)$

$$= \frac{1}{6}\left[(2-x)^3(-24) + (x-1)^3(276)\right] + (2-x)\left[2 - \frac{1}{6}(-24)\right] + (x-1)\left[33 - \frac{1}{6}(276)\right]$$

$$= \frac{1}{6}\left[(8 - 12x + 6x^2 - x^3)(-24) + (x^3 - 3x^2 + 3x - 1)(276)\right] + 6(2-x) - 13(x-1)$$

$$= 50x^3 - 162x^2 + 167x - 53$$

on $[2,3]$: $F(x) = \frac{1}{6}\left[(3-x)^3 M_2 + (x-2)^3 M_3\right] + (3-x)\left(f_2 - \frac{1}{6}M_2\right) + (x-2)\left(f_3 - \frac{1}{6}M_3\right)$

$$= \frac{1}{6}\left[(27 - 27x + 9x^2 - x^3)(276)\right] + (3-x)\left[33 - \frac{1}{6}(276)\right] + (x-2)[244]$$

$$= -46x^3 + 414x^2 - 985x + 715$$

an estimate at 2.5 is

$f(2.5) = F_3(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25$

Ans

Euler's Method :

Consider the scalar, first order initial value problem.

$$\left.\begin{array}{l} y'(t) = f(t, y(t)), \quad a \le t \le b \\ y(a) = \alpha \end{array}\right] \text{——} (1)$$

Let $y(t)$ be exact solⁿ of eqⁿ (1) and w be approximate solⁿ of eqⁿ (1) i.e $y \approx w$.

The interval $[a, b]$ divided into $(n+1)$ distinct pts i.e

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$$

w_i represent the approximate solⁿ to $y_i = y(t_i)$.

$$h = \frac{(b-a)}{N} \quad \text{step length.}$$

$$t_i = a + ih \quad (i = 0, 1, 2, N)$$

Let $y(t)$ has two cts derivatives. Expanding this true solⁿ in taylor series about the point $t = t_i$ produces.

$$y(t) = y(t_i) + (t - t_i) y'(t_i) + \frac{1}{2} (t - t_i)^2 y''(\xi)$$

$$\text{or} \quad y(t) = y_i + (t - t_i) y_i' + \frac{1}{2} (t - t_i)^2 y''(\xi) \quad \text{——} (2)$$

where $t \le \xi \le t_i$

at $t = t_{i+1}$, from eqⁿ (2) we get

$$y(t_{i+1}) = y_i + (t_{i+1} - t_i) y_i' + \frac{1}{2} (t_{i+1} - t_i)^2 y''(\xi)$$

$$y_{i+1} = y_i + h f(t_i, y_i) + \frac{1}{2} h^2 y''(\xi)$$

or
$$\boxed{\begin{array}{l} y_{i+1} = y_i + h f(t_i, y_i) \\ y_0 = \alpha \end{array}}$$
dropping the Error term.

Now replacing y_i by w_i

$$\boxed{\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\, f(t_i, w_i), \quad i = 0, 1, 2, \ldots N-1 \end{aligned}}$$

This is known as Euler's method for ODEs (IVP).

<u>Example</u>: Solve the IVP

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \le t \le 6 \qquad \text{—— (1)}$$

$$x(1) = 1 \qquad N = 10$$

by Euler method.

<u>Sol</u>^{n}: Here $f(t, x) = 1 + \frac{x}{t}$, $\quad a = 1, \; b = 6, \; N = 10$

step length $h = \frac{b-a}{N} = \frac{6-1}{10} = \frac{5}{10} = 0.5$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots N-1$$

By Euler method,

$$w_{i+1} = w_i + h\, f(t_i, w_i)$$

$$\boxed{\begin{aligned} w_{i+1} &= w_i + h\left(1 + \frac{w_i}{t_i}\right) \\ w_0 &= \alpha = 1 \end{aligned}} \qquad \text{—— (2)}$$

for $i = 0$, $\quad w_1 = w_0 + h\left(1 + \frac{w_0}{t_0}\right)$

$$w_1 = 1 + 0.5\left(1 + \frac{1}{1}\right)$$

$$= 1 + 0.5(2)$$

$$\boxed{w_1 = 2}$$

for $i=2$: $\quad w_2 = w_1 + h\left(1+\frac{w_1}{t_1}\right)$

where $t_1 = t_0 + h = 1 + 0.5 = 1.5$

$$w_2 = 2 + 0.5\left(1 + \frac{2}{1.5}\right) = 3.166667$$

$$\boxed{w_2 = 3.1666667}$$

Continuing in this fashion, we obtain the following results, with the exact value of the exact solⁿ $x(t) = t(1+\ln t)$ listed for Comparison.

t	Approx. solⁿ.	Exact solⁿ.	Abs Error $= \lvert y(t_i) - w_i \rvert$
1.0	1	1	0
1.5	2	2.10819766	0.108198
2	3.1666667	3.38629436	0.219628
2.5	4.45833333	4.79072683	0.332393
3.0	5.8500000	6.29583687	0.445837
3.5	7.32500000	7.88467039	0.559670
4.0	8.87142857	9.54577744	0.673749
4.5	——		
5	——		
5.5	——		
6	——		

Ans.

EXERCISES

For Exercises 1–6, apply Euler's method to approximate the solution of the given initial value problem over the indicated interval in t using the indicated number of time steps.

1. $x' = tx^3 - x \quad (0 \le t \le 1), \quad x(0) = 1, \quad N = 4$
2. $x' + (4/t)x = t^4 \quad (1 \le t \le 3), \quad x(1) = 1, \quad N = 5$
3. $x' = (\sin x - e^t)/\cos x \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 3$
4. $x' = (1 + x^2)/t \quad (1 \le t \le 4), \quad x(1) = 0, \quad N = 5$
5. $x' = t^2 - 2x^2 - 1 \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 4$
6. $x' = 2(1 - x)/(t^2 \sin x) \quad (1 \le t \le 2), \quad x(1) = 2, \quad N = 3$

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(I) Heun's Method : Let $y' = f(x, y)$, $a \le x \le b$ (IVP).

$$\boxed{y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)}$$

Where $K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h, y_n + K_1)$

$h \rightarrow$ step length.

(II) Modified Euler - Cauchy Method:

$$\boxed{y_{n+1} = y_n + K_2}$$

Where $K_1 = h f(x_n, y_n)$.

$K_2 = h f(x_n + \frac{h}{2}, y_n + \frac{1}{2}K_1)$.

Example: Find the approximate value of $y(1.4)$ for the IVP

$y' = x^2 + y^2$, $y(1) = 2$

with $h = 0.2$, using the (I) Heun's method (II) modified Euler - Cauchy method.

Soln: We have $f(x, y) = x^2 + y^2$, $h = 0.2$

(I) Heun's method is given by

$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$

$K_1 = h f(x_n, y_n)$, $K_2 = h f(x_n + h, y_n + K_1)$

We have the following results.

$n=0, \quad x_0=1, \quad y_0=2.$

$k_1 = h f(x_0, y_0) = 0.2 f(1,2) = 1$

$k_2 = h f(x_0+h, y_0+k_1) = 0.2 f(1.2, 3) = 2.088$

$y(1.2) = y_1 = y_0 + \frac{1}{2}(k_1+k_2) = 2+\frac{1}{2}(1+2.088) = 3.544.$

$\underline{n=1}:$ $x_1 = 1.2, \quad y_1 = 3.544$

$k_1 = h f(x_1, y_1) = 0.2 f(1.2, 3.544) = 2.8$

$k_2 = h f(x_1+h, y_1+k_1) = 0.2 f(1.4, 6.344) = 8.4413.$

$y(1.4) = y_2 + \frac{1}{2}(k_1+k_2) = 3.544 + \frac{1}{2}(2.8 + 8.4413)$

$= \underline{9.1647}.$

(ii) Modified Euler-Cauchy method is given by

$y_{n+1} = y_n + k_2$

$k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1)$

we have the following result

for $n=0$, $x_0 = 1$, $y_0 = 2$, $h = 0.2$

$k_1 = h f(x_0, y_0) = 0.2 f(1,2) = 1$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1) = 0.2 f(1.1, 2.5) = 1.492$

$y(1.2) = y_1 = y_0 + k_2 = 3.492.$

for $\underline{n=1}$ $x_1 = 1.2, \quad y_1 = 3.492.$

$$k_1 = h\, f(x_1, y_1) = 0.2\, f(1.2,\ 3.492).$$
$$= 2.7268$$
$$k_2 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{1}{2}k_1\right)$$
$$= 0.2\, f(1.3,\ 4.8554)$$
$$= 5.0530$$
$$y(1.4) \approx y_2 = y_1 + k_2 = 3.492 + 5.0530$$
$$= \underline{\underline{8.545}}\,.$$

Mid-point Method:

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$$y_{n+1} = y_{n-1} + 2h f(x_n, y_n), \quad n = 0, 1, 2, \ldots N$$

where h be step length.

Ex: Solve the initial value problems

$$y' = -2xy^2, \quad y(0) = 1$$

using the mid point method, with $h = 0.2$ over the interval $[0, 0.6]$.

Sol^n: we have $f(x, y) = -2xy^2$

step length $h = 0.2$.

The mid-point method

$$y_{n+1} = y_{n-1} + 2h\, f(x_n, y_n)$$

or $$y_{n+1} = y_{n-1} + 2h\,(2x_n y_n^2)$$

or $$y_{n+1} = y_{n-1} + 4h\, x_n y_n^2 \quad \text{——} (1)$$

for $n = 1$

$$y_2 = y_0 + 4h\, x_1 y_1^2 \quad \text{——} (2)$$

initial we have $x_0 = 0$, $y_0 = 1$, $x_1 = x_0 + h = 0.2$

we calculate y_1 from the Taylor series method, we have

$$y_1 \simeq y(h) = y(0) + h\, y'(0) + \frac{h^2}{2} y''(0)$$

where $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

we obtain $$y_1 \simeq y(0.2) = 1 + 0 + \frac{0.04}{2}(-2) = 0.96.$$

hence from eq^n ② we get

$$y_2 = 1 + 4(0.2)(0.2)(.96)^2$$

$y(x_0+2h) = y_2 = 0.852544$

or $y(0.4) = y_2 = 0.852544$

$$\boxed{y_2 = 0.852544}$$

for $n=2$: from eq^n ① we get

$$y_3 = y_1 + 4h\, y_2^2 x_2$$

$x_2 = x_0 + 2h$

$x_2 = 0 + 2 \times 0.2$

$\boxed{x_2 = 0.4}$

$\cancel{y_3 = 0.852544 + 4(0.2) \times 0.4 \times (0.852544)^2}$

$y_3 =$

$$y_3 = 0.96 - (0.8) \cdot (0.4)(0.852544)^2$$

$y(x_0+3h) = y_3 = 0.7274139.$

$y(0+3\times 0.2) = y_3 = 0.7274139$

$y(0.6) = 0.7274139.$

Q1. Apply Euler's modified method to approximate the solution of the IVP and calculate $y(1.3)$ by using $h=0.1$.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \quad y(1)=1.$$

Q.2: Apply mid-point method (R.K. Second order) to solve the IVP.

$$\frac{dy}{dx} = yx^3 - 1.5y$$

from $x=0$ to 2 where $y(0)=1$ by using $h=1$.

Dated: 20/03/2020

Runge-Kutta method:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad n = 0, 1, 2, ..$$

where
$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1)$$
$$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2)$$
$$k_4 = h f(x_n + h, y_n + k_3)$$

This is classical Runge-Kutta method of fourth order.

Example: Solve the initial value problem

$$y' = x(y - x), \quad y(2) = 3$$

in the interval $[2, 2.4]$ using the classical Runge-kutta fourth order method with the step length $h = 0.2$.

Sol^n: we have $f(x, y) = x(y - x)$, $h = 0.2$

for $n = 0$, $x_0 = 2$, $y_0 = 3$

$$k_1 = h f(x_0, y_0) = 0.2 f(2, 3) = 0.4$$
$$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1) = 0.2 f(2.1, 3.2) = 0.4620$$
$$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2) = 0.2 f(2.1, 3.231) = 0.4750$$
$$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(2.2, 3.4750) = 0.5610$$

by R.K method

$$y(2.2) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 3 + \frac{1}{6}[0.4 + 2(0.4620 + 0.4750) + 0.5610] = 3.4725$$

for $n=1$, $x_1 = 2.2$, $y_1 = 3.4725$

$k_1 = hf(x_1, y_1) = 0.2\, f(2.2, 3.4725) = 0.5599$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1) = 0.2\, f(2.3, 3.7525)$

$= 0.6682$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2) = 0.2 f(2.3, 3.8066)$

$= 0.6930$

$k_4 = hf(x_1 + h, y_1 + k_3) = \cancel{0.2 f(2.3, 3.8066) = 0.6930}$

$= 0.2\, f(2.4, 4.1655)$

$= 0.8474$

by R.K method,

$y(2.4)$ $y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 3.4725 + \frac{1}{6}[0.5599 + 2(0.6682 + 0.6930)$

$+ 0.8474] = 4.1608.$

Ans.

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Exercise 18.7

In the following initial value problems, find the approximate values of $y(x)$ at the given points using the Euler method.

1. $y' = xy + x^2y^2 + 1$, $y(1) = 2$, $h = 0.1$, $x \in [1, 1.3]$.
2. $y' = 3x^2 + \sqrt{y}$, $y(1) = 1$, $h = 0.2$, $x \in [1, 2]$.

In the following initial value problems, find the approximate values of $y(x)$ at the given points using the Taylor series method of given order.

3. $y' = 1 + y^2$, $y(0) = 1$, $h = 0.1$, order $= 2$, $x \in [0, 0.3]$.
4. $y' = x^2 + \sqrt{y}$, $y(1) = 1$, $h = 0.1$, order 3, $x \in [1, 1.2]$.
5. $y' = x + \sin y$, $y(1) = 0$, $h = 0.2$, order $= 4$, at $x = 1.4$.

In the following initial value problems, find the approximate values of $y(x)$ at the given points using Heun's method.

6. $y' = x^2 + 2y^2$, $y(1) = 1$, $h = 0.2$, $x \in [1.0, 1.4]$.
7. $y' = \dfrac{y - x}{y + x}$, $y(1) = 2$, $h = 0.1$, $x \in [1.0, 1.2]$.

In the following initial value problems, find the approximate values of $y(x)$ at the given points using classical fourth order Runge-Kutta method.

8. $y' = x^2 + y^2$, $y(1) = 2$, $h = 0.1$, $x \in [1.0, 1.2]$.
9. $y' = \sqrt{x + y}$, $y(0.4) = 0.41$, $h = 0.2$, $x \in [0.4, 0.8]$.
10. $y' = 2 + \sqrt{xy}$, $y(2) = 1$, $h = 0.1$, $x \in [2.0, 2.2]$
11. Find an approximate value of ...

Dated 01/04/2020

Finite difference Method:

We consider the linear second order diff. eq^n

$$-y'' + p(x)\,y' + q(x)\,y = r(x), \qquad a < x < b \qquad \text{--- (1)}$$

subject to the boundary conditions of the first kind

$$y(a) = \gamma_1, \quad y(b) = \gamma_2$$

We will solve problem (1) by following steps.

Step 1: The nodal points on an interval $[a, b]$ may be defined by

$$x_i = x_0 + ih, \quad i = 1, 2, \ldots, N+1$$

or $$x_i = a + ih, \quad i = 1, 2, \ldots, N+1$$

where $x_0 = a$, $x_{N+1} = b$, and $h = \dfrac{b-a}{(N+1)}$

Step 2: We know that the approximate value of y' and y'' at x_i point

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_{i-1})}{2h}$$

or $$\boxed{y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}}$$

and $$y''(x_i) \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i+1})}{h^2}$$

or $$\boxed{y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}}$$

Step 3: The finite difference approximation of the diff. eq^n (1) at $x = x_i$ is given by

$$-\left[\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right]+p(x_i)\left[\frac{y_{i+1}-y_{i-1}}{2h}\right]+q(x_i)y_i=r(x_i)$$

$$i=1,2,\dots N \quad \text{——— (2)}$$

The boundary conditions becomes

$$y_0=\gamma_1,\quad y_{N+1}=\gamma_2$$

Multiplication by $\frac{h^2}{2}$ in eqn (2) may be written in the form.

$$A_i\,y_{i-1}+B_i\,y_i+C_i\,y_{i+1}=\frac{h^2}{2}\,r(x_i),\quad i=1,2,\dots N \quad \text{——— (3)}$$

where

$$A_i=-\frac{1}{2}\left(1+\frac{h}{2}\,p(x_i)\right)$$

$$B_i=\left(1+\frac{h^2}{2}\,q(x_i)\right)$$

$$C_i=-\frac{1}{2}\left(1-\frac{h}{2}\,p(x_i)\right)$$

Step.4: The system (3) in matrix notation, after incoporating the boundary conditions becomes, ——— (4)

$$Ay=b$$

where

$$y=[y_1,y_2,\dots,y_N]^T$$

$$b=\frac{h^2}{2}\left[r(x_1)-\frac{2A_1\gamma_1}{h^2},\ r(x_2),\dots,r(x_{N-1}),\ r(x_N)-\frac{2C_N\gamma_2}{h^2}\right]^T$$

$$A=\begin{bmatrix} B_1 & C_1 & & & 0 \\ A_2 & B_2 & C_2 & \dots & \vdots \\ \dots & \dots & \dots & & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & - & \dots & -A_N & B_N \end{bmatrix}$$

The sol^n of the system of linear eq^n (4) gives the finite difference sol^n of the diff. eq^n (1) satisfying the boundary conditions.

Example: Solve the boundary value problem

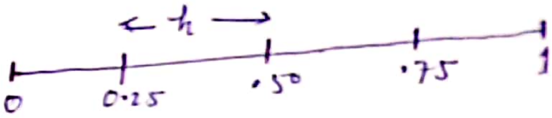
$$y'' = y + x \qquad \text{—— (1)}$$

$$y(0) = 0, \quad y(1) = 0 \qquad \text{with } h = \frac{1}{4}$$

by finite difference method.

Sol^n: We divide the interval $[0,1]$ into four subintervals. and $h = 0.25$.

The nodal points are $x_i = x_0 + ih$, $i = 0,1,2,3,4$



We know that the approximate value of y'' at x_i point is

$$y''(x_i) \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

The finite difference approximation of the diff. eq^n (1) at a point $x = x_i$.

by putting value of $y''(x_i)$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = y_i + x_i, \quad i = 1,2,3.$$

multiplying by $-h^2$ we obtain.

$$-y_{i-1} + 2y_i - y_{i+1} = -h^2(y_i + x_i), \quad i = 1,2,3.$$

$$-y_{i-1} + 2y_i - y_{i+1} = -\left(\frac{1}{4}\right)^2 (y_i + x_i)$$

$$-16\, y_{i-1} + 33\, y_i - 16\, y_{i+1} = -x_i$$

we have ,

for $i=1$: $\quad -16\, y_0 + 33\, y_1 - 16\, y_2 = -x_1$

or $\quad -16\, y_0 + 33\, y_1 - 16\, y_2 = -0.25$

for $i=2$: $\quad -16\, y_1 + 33\, y_2 - 16\, y_3 = -0.50$

for $i=3$: $\quad -16\, y_2 + 33\, y_3 - 16\, y_4 = -0.75$

using the boundary conditions $y_0 = y_4 = 0$ we get the system of eq^{n}s.

$$33 y_1 - 16 y_2 = -0.25$$
$$-16 y_1 + 33 y_2 - 16 y_3 = -0.50$$
$$-16 y_2 + 33 y_3 = -0.75$$

The above system can be written as in matrix form.

$$\begin{bmatrix} 33 & -16 & 0 \\ -16 & 33 & -16 \\ 0 & -16 & 33 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = - \begin{bmatrix} 0.25 \\ 0.50 \\ 0.75 \end{bmatrix}$$

which gives $\quad y_1 = -0.034885, \quad y_2 = -0.056326,$

$y_3 = -0.050037$

Ans.

Assignment:

Q.1. ~~Find~~ Apply finite difference method to solve the problem.

$$\frac{d^2y}{dx^2} = y + x, \quad y(0) = 2, \quad y(1) = 2.5 \quad \text{with } h = 0.25.$$

Q.2: Apply finite difference method to solve the given problem.

$$\frac{d^2y}{dx^2} = y + x(x-4), \quad 0 \le x \le 4$$

$$y(0) = 0, \quad y(4) = 0, \quad \text{with } h = 1.$$

Newton's forward difference Interpolation:

We know that the divided difference

Ist d.d., $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{1}{h}\Delta f_0$

$h \to$ step length.
$x_i = x_0 + ih$, $i = 0, 1, 2 \cdots n$

IInd d.d., $f[x_0, x_1, x_2] = \frac{1}{2!h^2}\Delta^2 f_0$

$\vdots$

n^{th} d.d., $f[x_0, x_1, \cdots x_n] = \frac{1}{n!\,h^n}\Delta^n f_0$

Replacing divided difference by forward differences in the divided difference interpolating poly., we get

$$\boxed{P_n(x) = f_0 + (x-x_0)\frac{\Delta f_0}{1!\,h} + (x-x_0)(x-x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}}$$

Which is called Newton's forward diff. int. poly.

Newton's Backward difference Int. poly.:

$$\boxed{P_n(x) = f_n + (x-x_n)\frac{\nabla f_n}{1!\,h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!\,h^2} + \cdots + (x-x_n)(x-x_{n-1})\cdots(x-x_1)\frac{\nabla^n f_n}{n!\,h^n}}$$

$\Delta \to$ forward difference

$\nabla \to$ backward difference

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Example : For the data

x	-4	-2	0	2	4	6
$f(x)$	-139	-21	1	23	141	451

Construct the forward and backward difference Tables. Using the Corresponding interpolation, show that the int. poly. is same.

Sol^n: The step length $h = 2$, and $x_0 = -4$, $x_1 = -2$, $x_2 = 0$
$x_3 = 2$, $x_4 = 4$, $x_5 = 6$.

We have the following difference Table

x	$f(x)$	$\Delta f / \nabla f$	$\Delta^2 f/\nabla^2 f$	$\Delta^3 f/\nabla^3 f$	$\Delta^4 f/\nabla^4 f$	
$-4 = x_0$	$-139 = f_0$					
		118 (Δf_0)				
$-2 = x_1$	$-21 = f_1$		-96 ($\Delta^2 f_0$)			
		22		96 ($\Delta^3 f_0$)		
$0 = x_2$	$1 = f_2$		0		0 ($\Delta^4 f_0$)	
		22		96		0 ($\Delta^5 f_0$, $\nabla^5 f_5$)
$2 = x_3$	$23 = f_3$		96		0 ($\nabla^4 f_5$)	
		118		96 ($\nabla^3 f_5$)		
$4 = x_4$	$141 = f_4$		192 ($\nabla^2 f_5$)			
		310 (∇f_5)				
$6 = x_5$	$451 = f_5$					

Note
$$\Delta f_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-139+21}{-2+4} = -96$$
$$\nabla f_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = -96.$$

-itude dependent terms in the

Using the Newton's forward diff. interpolating poly.

$$P_3(x) = f_0 + (x-x_0)\frac{\Delta f_0}{1!\,h} + (x-x_0)(x-x_1)\frac{\Delta^2 f_0}{2!\,h^2} +$$

$$(x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f_0}{3!\,h^3} + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\times\frac{\Delta^4 f_0}{4!\,h^4}$$

$$= -139 + (x+4)\frac{118}{2} + (x+4)(x+2)\frac{(-96)}{4\cdot(2)^2} +$$

$$(x+4)(x+2)x\frac{96}{6\cdot(2)^3} + (x+4)(x+2)(x)(x-2)\times 0$$

$$= -139 + 59(x+4) - 12(x+4)(x+2) + 2x(x+4)(x+2)$$

$$= 2x^3 + 3x + 1$$

Using the Newton's backward diff. Int.

(b)

$$P_3(x) = f_5 + (x-x_5)\frac{\nabla f_5}{1!\,h^5} + (x-x_5)(x-x_4)\frac{\nabla^2 f_5}{2!\,h^2} +$$

$$(x-x_5)(x-x_4)(x-x_3)\frac{\nabla^3 f_5}{3!\,h^3} + (x-x_5)(x-x_4)(x-x_3)(x-x_3)\frac{\nabla^4 f_5}{4!\,h^4} +$$

$$(x-x_5)(x-x_4)(x-x_3)(x-x_2)(x-x_1)\frac{\nabla^5 f_5}{5!\,h^5}$$

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$$P_3(x) = 451 + (x-6)\left[\frac{310}{2}\right] + (x-6)(x-4)\left[\frac{192}{2\times 4}\right]$$

$$+ (x-6)(x-4)(x-2)\left[\frac{96}{6\times 8}\right]$$

$$= 451 + 155(x-6) + 24(x-6)(x-4) + 2(x-6)(x-4)(x-2)$$

$$= 2x^3 + 3x + 1$$