Tests for the Convergence of a +ve term Series

① Comparison test

①(a) If $\sum a_n$ & $\sum b_n$ be two +ve term series

such that $a_n < K b_n$ where $K > 0$

$\forall n \geq m$

then if (i) $\sum b_n$ is a cgt series

then $\sum a_n$ will also be cgt series.

(ii) if $\sum a_n$ is a div. series

$\Rightarrow \sum b_n$ is also a div. series

( i.e. $a_n < K b_n$ ) cgt div.

①(b) If $\sum a_n$ & $\sum b_n$ be two +ve term series such that

$k b_n < a_n < K b_n$ where $k$ & $K$ are +ve real nos.

then $\sum a_n$ & $\sum b_n$ will Converge or diverge together.

Exp. ① Test for the Convergence of the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\log n} = \frac{1}{\log 2} + \frac{1}{\log 3} + \dots$$

here $a_n = \frac{1}{\log n}$

& $b_n = \frac{1}{\log n}$

we know that $\log n < n \ \forall$

$\Rightarrow \frac{1}{n} < \frac{1}{\log n}$

( i.e. $a_n < K b_n$ where $K = 1$

we have $\frac{1}{n} < \frac{1}{\log n}$ (which is of the form $a_n < k\, b_n$ where $\sum a_n$ is a div. series Page (25) )

$\therefore$ by Comparison test (1)(a) we get $\sum \frac{1}{\log n}$ is a div. Series.

Exp. (2) Test for the Convergence of the series

$$\sum a_n = \sum \frac{1}{n^n} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots$$

here $a_n = \frac{1}{n^n}$

we know that $n^n > 2^n$ (let $n > 2$)

$$\Rightarrow \frac{1}{n^n} < \frac{1}{2^n} \quad (*)$$

(which is of the form $a_n < k\, b_n$ $n \geq 2$ where $a_n = \frac{1}{n^n}$, $k = 1$, $b_n = \frac{1}{2^n}$ )

cgt $a_n < k\, b_n$

Also, we know that

$\sum \frac{1}{2^n}$ is cgt series

( $\because$ it is a geometric series with common ratio $= \frac{1}{2}$ which is less than '1' )

$\therefore$ by comparison test (1)(a) we get $\sum a_n = \sum \frac{1}{n^n}$ is cgt series

→ **Integral test** (without Pf)

If a fn. $f(x)$ is a non-neg., monotonically decreasing & integrable fn. (for $x \geq 1$)

Such that $f(n) = a_n \quad \forall n \in \mathbb{N}$

then the series $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$

Converge or diverge together.

**Application** :

Show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is cgt Series if $p > 1$ $(p \in \mathbb{R})$

(ie. Series of the form $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$ etc. are cgt series.)

Solⁿ: Let $f(x) = \frac{1}{x^p} \quad x \geq 1, \; p > 1$

then $f$ is a non-neg, monotonically dec. integrable fn.

Such that
$$f(n) = \frac{1}{n^p}$$

$\therefore$ by Integral test $\sum \frac{1}{n^p}$ & $\int_1^{\infty} \frac{1}{x^p}\,dx$ will Converge or diverge together.

Now T.S. $\int_1^{\infty} \frac{1}{x^p}\,dx$ is cgt for $p > 1$

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we have

$$\int_1^{\infty} f(x)\,dx = \underset{X\to\infty}{Lt} \int_1^{X} f(x)\,dx$$

$$= \underset{X\to\infty}{Lt} \int_1^{X} \frac{1}{x^p}\,dx \qquad p>1$$

$$= \underset{X\to\infty}{Lt} \left[\int_1^{X} x^{-p}\,dx\right]$$

$$= \underset{X\to\infty}{Lt} \left[\left[\frac{x^{-p+1}}{-p+1}\right]_1^{X}\right]$$

$$= \underset{X\to\infty}{Lt} \left[\frac{X^{1-p}}{(1-p)} - \frac{1}{(1-p)}\right] \text{ --- } (*)$$

Since $p>1$

$\Rightarrow 1-p<0 \quad \Rightarrow X^{1-p} \longrightarrow 0$ as $X\to\infty$

$\therefore\ (*) = -\frac{1}{(1-p)} = \frac{1}{p-1}$

$\Rightarrow \int_1^{\infty} f(x)\,dx = \frac{1}{p-1} \quad \therefore \int_1^{\infty} f(x)\,dx$ is cgt

$\Rightarrow \sum \frac{1}{x^p}$ is a cgt series for $p>1$

→ Show that $\sum \frac{1}{n}$ is a div. series, using Integral test

[Hint : $\because \int_1^{\infty} \frac{1}{x}\,dx$ is a div. Integral

(Integrate & check

$p>0$

→ Show that the series $\sum \frac{1}{n^p}$ where $0<p<1$ is a div. series

(i.e. the series of the form $\sum \frac{1}{\sqrt{n}}$, $\sum \frac{1}{n^{1/4}}$ ... etc.)

using Integral test

Now, $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{X\to\infty}\left[\int_1^X \frac{1}{x^p}\,dx\right]$

$= \lim_{X\to\infty}\left[\frac{X^{1-p}}{1-p} - \frac{1}{(1-p)}\right]$ — (A)

Since $p<1 \Rightarrow 1-p>0$ $\therefore X^{1-p}\to\infty$ as $X\to\infty$

$\therefore$ by (A) $\int_1^\infty \frac{1}{x^p}\,dx \to \infty$

$\Rightarrow \int_1^\infty \frac{1}{x^p}$ is a div. Integral for $p<1$

$\Rightarrow \sum \frac{1}{n^p}$ for $p<1$, is a div. series

Conclusion

$\sum \frac{1}{n^p}$ (Called p-Series)

is a cgt series if $p>1$

& is a div. series if $0<p\leq 1$

(Note : $p=1$ i.e. $\sum \frac{1}{n}$ is a div. series)

$\therefore$ $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$ etc are cgt series

whereas $\sum \frac{1}{\sqrt{n}}$, $\sum \frac{1}{n^{1/4}}$, ... etc. are div. series