→ Infinite Series (of Real Numbers)

Exp.

(*) $\sum u_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} + \cdots$

here $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ (finite sum of n terms)

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left[\frac{1}{2} - \frac{1}{3}\right] + \cdots + \left[\frac{1}{n} - \frac{1}{n+1}\right]$

$= 1 - \frac{1}{n+1} \to 1$ as $n \to \infty$

ie. (S_n), the seq. of partial sums of the given series $\sum u_n$ is cgt & converges to '1'

$\therefore$ $\sum u_n$, the series also converges & its sum = 1

Note: (*); this series is an exp. of a telescoping series.

Defn: A series is called a telescoping series in which all the terms except a few terms get cancelled.

→ Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \qquad a \neq 0.$$

A geometric series is cgt iff $|r| < 1$

Pf: here $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (finite sum of n terms of the given series)

$$= \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \left(\frac{ar^n}{1-r}\right) \quad \text{——— (1)}$$

Now r^n converges iff $|r|<1$ (Seq. Concept)

$\therefore (S_n)$ converges iff $|r|<1$

i.e. SOPS of the given series is cgt iff $|r|<1$

$\therefore$ the given series is cgt iff $|r|<1$

Also, in case (S_n) is cgt. (i.e. when $|r|<1$), (S_n) converges to $\frac{a}{1-r}$ by ①

($\because r^n \to 0$ when $|r|<1$)

$\therefore$ the sum of the given geometric series

is $\frac{a}{1-r}$.

If $\sum_{n=1}^{\infty} a_n$ converges & if $c \neq 0$ then $\sum_{n=1}^{\infty} c a_n$ also converges

& $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ both converge with sum α & β respectively

then the series $\sum_{n=1}^{\infty} (a_n+b_n)$ also converges & its sum $= \alpha+\beta$.

→ A Necessary Condition for the Convergence of an infinite series.

(Also called n-th term test)

Thm: If $\sum_{n=1}^{\infty} a_n$ is a cgt series then its n-th term tends to zero

i.e. $a_n \to 0$ as $n \to \infty$.

Pf: here $S_n = a_1 + a_2 + \cdots + a_n = \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{\Rightarrow S_{n-1}} + a_n$

Since the given series is cgt (given)

$\therefore$ (S_n), the SOPS is a cgt Seq.

Let it converge to 'l' i.e. $S_n \to l$ as $n \to \infty$

Now, $S_n = S_{n-1} + a_n$

take $n \to \infty$

$\Rightarrow l = l + \lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} a_n = 0$

i.e. $a_n \to 0$ as $n \to \infty$.

Note: Converse of this thm may not be true.

Exp. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$, here $a_n = \ln\left(\frac{n}{n+1}\right)$

($\ln$ natural log)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\ln\left(\frac{n}{n+1}\right)\right)$

$= \ln\left(\lim \frac{n}{n+1}\right)$

$= \ln 1 = 0$

i.e. $a_n \to 0$ as $n \to \infty$

Also, S_n = Sum of first n terms of the given infinite series

$$= \ln\frac{1}{1+1} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1}$$

$$= \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1}$$

$$= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln(n+1))$$

$$= \ln 1 - \ln(n+1)$$

$$= 0 - \ln(n+1) = -\ln(n+1) \longrightarrow -\infty$$

$\therefore$ (S_n) is a div. seq.

$\therefore$ $\sum_{n=1}^{\infty} a_n$ is a div. series.

Remark: This thm is more useful for the series $\sum_{n=1}^{\infty} a_n$ when $a_n \not\to 0$ as $n \to \infty$

$\because$ in that case the given series will not be cgt,

because if the series is cgt then its nth term must tend to zero.

Note: So, to check the convergence or div. of the given series, first of all we check its nth term, if nth term doesn't tend to zero then the given series is div.

Exp. Consider the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + (\cancel{\quad})\frac{n}{n+1} + \cdots$$

here $a_n = (\cancel{-1})^{n+1} \cdot \frac{n}{n+1} \not\to 0$ as $n \to \infty$

$\therefore$ the given series is not cgt.

(by nth term test)

Ex.

Determine if the following series converge or diverge using only the defn (i.e. using SOPS method)

(i) $\sum_{n=1}^{\infty} a_n = 1+2+3+\cdots\cdots$

Soln: here S_n = SOPS of the given series $\sum_{n=1}^{\infty} a_n$

$= 1+2+3+\cdots+n$ (Sum of first n terms of the given series)

$= \frac{n(n+1)}{2}$

$\longrightarrow \infty$ as $n \to \infty$

$\Rightarrow (S_n)$ is a div. seq.

$\Rightarrow \sum a_n$ is a div. series

(ii) $\sum_{n=1}^{\infty} a_n = 1-3+5-7+\cdots$

here $S_1 = 1$

$S_2 = 1-3 = -2$

$S_3 = 1-3+5 = 3$ etc.

$S_4 = 1-3+5-7 = -4$

$\therefore S_{2n+1} = (2n+1)$
$\& \ S_{2n} = -2n$ $\Big\} \Rightarrow (S_n)$ is div se

i.e. SOPS of the giv Series is a div.

$\therefore$ the given Serie is a div. Seri

(III) $1+1-1-1+1+1-1-1+\cdots\cdots$

(do it yourself)

(iv) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Telescoping Se

here $S_n = \left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\cdots+($

$= 1-\frac{1}{\sqrt{n+1}} \to 1$ as $n \to \infty$

$\Rightarrow (S_n)$, SOPS of the given seri i & Converges to 1

$\therefore$ the given series is Convergent & its Sum $= 1$

(V) $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ (telescoping Series)

$= \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{1}{(2n-1)}-\frac{1}{(2n+}\right.$

.... Complete the Sol^n (Ex