Exp. Show that the series $\sum (-1)^{n+1} n e^{-n}$ is a cgt series.

Sol^n: we have

$$\sum (-1)^{n+1} a_n = \sum (-1)^{n+1} n e^{-n}$$

here $a_n = n e^{-n}$ (T.S. $a_{n+1} < a_n \ \forall n$ & $a_n \to 0$ as $n \to \infty$)

Let $f(x) = x e^{-x}$

then $f'(x) = x(-e^{-x}) + e^{-x} = e^{-x}(1-x)$

$$= \frac{(1-x)}{e^x} \leq 0 \ \forall x \geq 1$$

$\therefore$ $f(x)$ is a dec. fn. $\Rightarrow f(n+1) < f(n) \ \forall n$

Now $f(n) = n e^{-n} = a_n$ (as $n+1 > n$ $\therefore f(n+1) < f(n)$)

& $f(n+1) = a_{n+1} \quad \forall n$

hence $a_{n+1} < a_n \ \forall n$ — (1)

Now T.S. $a_n \to 0$ as $n \to \infty$ ($\frac{\infty}{\infty}$ form)

Since $\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

$\Rightarrow \lim_{n \to \infty} n e^{-n} = 0 \Rightarrow a_n \to 0$ as $n \to \infty$ — (2)

by (1) & (2), the given series $\sum (-1)^{n+1} n e^{-n}$ is a cgt series (Leibniz test)

Ex Show that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots \text{ Converges.}$$

Sol^n: Since the given series is an alternating series, we will use Leibniz test to show that it is a cgt series.

Date:
Page No.

here $a_n = \frac{\log n}{n^2}$ $n \geq 2$

we know that $\frac{\log x}{x^2} \to 0$ as $x \to \infty$ (Show!)

($\frac{\infty}{\infty}$ form)

$\therefore$ $a_n \to 0$ as $n \to \infty$ — (1)

T.S. $a_{n+1} < a_n$ $\forall n$

Consider $f(x) = \frac{\log x}{x^2}$

$\Rightarrow f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \log x\,(2x)}{x^4} = \frac{x - 2x\log x}{x^4}$

$= \frac{1 - 2\log x}{x^3}$

< 0 if $2\log x > 1$

i.e. if $\log x > \frac{1}{2}$

or if $x > e^{1/2}$

$\therefore$ $f(x)$ is a dec. fn. if $x > e^{1/2} > 1$

$\Rightarrow f(x)$ is a dec. fn. $\forall x > 1$

$\Rightarrow f(n) > f(n+1)$ $\forall n \geq 2$

$\Rightarrow a_n > a_{n+1}$ $\forall n \geq 2$ — (2)

by (1) & (2), The given alternating Series is a cgt Series (Leibniz t

(1) Test for Convergence (Do it!)

(i) $\sum (-1)^{n+1} \cdot \frac{1}{(2n-1)}$

(ii) $\sum (-1)^{n} \cdot \frac{n}{n^2+1}$

(iii) $\sum (-1)^{n+1} \cdot \frac{1}{n^2}$

(iv) $\sum_{n=2}^{\infty} (-1)^{n} \cdot \frac{1}{\ln n}$

Absolute Convergence & Conditional Convergence.

A Series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely cgt if $\sum_{n=1}^{\infty} |a_n|$ converges

Exp : The Series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots = \sum (-1)^{n+1}\cdot\frac{1}{n^2}$

is absolutely cgt $\because \sum \left|(-1)^{n+1}\cdot\frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ is a cgt Series.

whereas $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots = \sum (-1)^{n+1}\cdot\frac{1}{n}$ is not an absolutely cgt Series

$\because \sum \left|(-1)^{n+1}\cdot\frac{1}{n}\right| = \sum \frac{1}{n}$ is a div. Series

(However, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is a cgt Series by Leibniz test)

Such type of Series is called a Conditionally cgt Series which is a cgt Series but is not an absolutely cgt Series.

Thm : Every absolutely cgt Series is a cgt Series

Pf: We are given that $\sum |a_n|$ is a cgt Series

T.S. $\sum a_n$ is also a cgt Series. — (*)

We know that

$$a_n \leq |a_n| \quad \forall n \Rightarrow |a_n| - a_n \geq 0 \;\forall n \quad \text{—(1)}$$

$$\& \; -a_n \leq |a_n| \quad \forall n$$

$$\Rightarrow |a_n| - a_n \leq 2|a_n| \quad \forall n \quad \text{—(2)}$$

by (1) & (2)

$$0 \leq |a_n| - a_n \leq 2|a_n| \quad \forall n$$

Applying comparison test of +ve term series we get

$$\sum(|a_n| - a_n) \text{ will be a cgt series} \quad \text{—(3)}$$

$\because \sum 2|a_n|$ is given to be cgt. series

Now

$$\sum a_n = \sum(a_n - |a_n| + |a_n|)$$

$$= \sum |a_n| - (|a_n| - a_n)$$

will be cgt by (*) & (3)

$\therefore \sum a_n$ is cgt series.

$\therefore$ Abs. Convergence $\Rightarrow$ Convergence

Exp: Show that $\sum (-1)^n \cdot \frac{n}{n^3+1}$ is a cgt series.

Sol^n: Since $\left|(-1)^n \cdot \frac{n}{n^3+1}\right| = \frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$

& we know that $\sum \frac{1}{n^2}$ is cgt series

$\therefore$ by Comparison test, the given series is absolutely cgt & therefore cgt as well.

Ex (1) Show that the Series $\sum \frac{\sin n}{n^2}$ Converges

Sol^n; Since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$ & we know that $\sum \frac{1}{n^2}$ is a Cgt Series

$\therefore$ the given Series is absolutely Cgt & hence Cgt as well.

(2) $\sum (-1)^n e^{-n}$ (Check for Convergence)

Sol^n: Since $\left|(-1)^n e^{-n}\right| = e^{-n} = \frac{1}{e^n}$

Let $a_n = \frac{1}{e^n}$, then $a_n^{\frac{1}{n}} = \frac{1}{e} < 1$

$\therefore$ by Root test, the given Series is absolutely Cgt & hence it is cgt as well.

(3) $\sum (-1)^n \cdot \frac{\cos n\pi}{n}$ (Check for Convergence)

Sol^n We know that $\cos n\pi = (-1)^n$

$\therefore \sum (-1)^n \frac{\cos n\pi}{n} = \sum \frac{1}{n}$ which is a divergent Series,

(4) $\sum (-1)^n \cdot \frac{\sin n}{n^{3/2}}$ (

Sol^n: $\left|(-1)^n \cdot \frac{\sin n}{n^{3/2}}\right| \le \frac{1}{n^{3/2}}$, Apply Comparison test to & Show that it is abs. Cgt & hence Cgt.

Check for Convergence

(5) $\sum(-1)^n a_n = \sum(-1)^n \cdot \frac{n}{n^2+1}$, This is not absolutely cgt series
($\because$ as +ve term series it will behave like $\sum \frac{1}{n}$ which is a div. series)

To check for it's convergence we will apply Leibniz test

T.S. $a_n \to 0$ as $n \to \infty$
and $a_n > a_{n+1}$ $\forall n$

Now $a_n = \frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} \to 0$ as $n \to \infty$

$\therefore a_n \to 0$ as $n \to \infty$ — (1)

Also $f(x) = \frac{x}{x^2+1}$ is a dec. fn. for $x \geq 1$

$$\left[\because f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0 \ \forall x \geq 1\right]$$

$\therefore f(n+1) \leq f(n)$ $\forall n$ (as $n \geq 1$)

$\Rightarrow a_{n+1} \leq a_n$ $\forall n$ — (2)

by (1) & (2), the given alternating series is cgt
(by Leibniz test)

— o —