Ex: Test for Convergence

$$\sum a_n = \sum \frac{n^{n^2}}{(n+1)^{n^2}}$$

Sol^n: $a_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\therefore a_n^{\frac{1}{n}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+\frac{1}{n})^n}$$

Now, $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})^n}\right) = \frac{1}{e}$

Since, $l = \frac{1}{e} < 1$

$\therefore$ by root test, the given series is a cgt series

Ex: Show that the series
$\sum e^{-n^2}$ is a cgt series.

Sol^n: We know that $e^x > x \quad \forall x > 0$

$$\therefore e^{n^2} > n^2$$

$$\Rightarrow \frac{1}{n^2} > \frac{1}{e^{n^2}}$$

i.e. $\frac{1}{e^{n^2}} < \frac{1}{n^2}$, Since $\sum \frac{1}{n^2}$ is cgt series) (p-series with $p>1$)

$\therefore \sum \frac{1}{e^{n^2}}$ is cgt by Comparison test)

Ex Test for Convergence

$$\sum a_n = \sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$$

Sol^n: here $a_n = \frac{1}{(\log n)^n}$ $n>1$

$\therefore a_n^{\frac{1}{n}} = \frac{1}{\log n}$ $n>1$

Now, $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(\log n)} = 0$ $\therefore l=0$ which is less than 1

$\therefore$ by root test $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is a cgt Series.

EX :

① → Assuming $\sin\theta < \theta$ whenever $0<\theta<\pi/2$. Show that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ is cgt.. series

Sol^n: $\sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$ Apply Comparison test.

② → Show that $\sum \cos\left(\frac{1}{n}\right)$ is div. Series

Sol^n: Hint : $a_n \not\to 0$ as $n\to\infty$

③ → Show that $\sum \frac{1}{3^n+x}$ $\forall x>0$ is a cgt Series.

here $a_n = \frac{1}{3^n+x} < \frac{1}{3^n}$ $\forall x>0$, Apply Comparison test

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Alternating Series :

A Series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

$= a_1 - a_2 + a_3 - a_4 + \cdots$

is called an alternating Series where $a_n > 0$ $\forall n$

exp. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

is an alternating series.

Thm: (Alternating Series Test)
or
(Leibniz test) for the Convergence of an alternating series.

→ An alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is cgt

if (i) $a_{n+1} \leq a_n$ $\forall n$

(ii) $a_n \to 0$ as $n \to \infty$

(i.e. the terms are decreasing & nth term tends to zero as $n \to \infty$)

then the alternating series is a cgt Series.

Pf: The given Series is an alternating Series

$\sum (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

here $S_1 = a_1$

$S_2 = a_1 - a_2$

$S_3 = a_1 - a_2 + a_3$

$S_4 = a_1 - a_2 + a_3 - a_4$

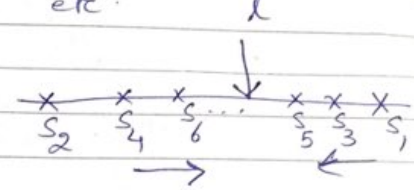
- - - - - etc.

Now $S_1 = a_1$

$S_2 = a_1 - a_2 \le a_1 \quad (\because a_2 \geq 0)$
$= S_1$

i.e. $S_2 \le S_1$

lly $S_4 \le S_3$, $S_6 \le S_5$ etc.



Also

$S_4 = \overbrace{a_1 - a_2} + a_3 - a_4$

$S_4 = S_2 + (a_3 - a_4)$
$\geq S_2 \quad (\because a_3 - a_4 \geq 0)$
given that the terms are dec.

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i.e. $S_{2n+2} \geq S_{2n}$

Similarly $S_5 = \overbrace{a_1 - a_2 + a_3} - a_4 + a_5$
$= S_3 - (a_4 - a_5) \le S_3 \quad (\because a_4 - a_5 \geq 0)$

i.e. $S_5 \le S_3$

$\therefore$ in general $S_{2n+1} \le S_{2n-1}$

So, we have

$S_2 \le S_4 \le S_6 \le \cdots$

& $S_1 \geq S_3 \geq S_5 \geq \cdots$

be two sequences where $(S_{2n}) = (S_2, S_4, \ldots)$ is $\uparrow$ + bdd by S_1

$(S_{2n-1}) = (S_1, S_3, S_5, \ldots)$ is dec + bdd by S_2

$\therefore$ both of them Converge. (by Monotone convergence thm)

Let $(S_{2n}) \to \alpha$

& $(S_{2n-1}) \to \beta$

$$S_{2n} = \overbrace{a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1}}^{S_{2n-1}} - a_{2n}$$

Since $S_{2n} = S_{2n-1} - a_{2n}$

$\Rightarrow \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n-1} - \lim_{n\to\infty} a_{2n}$

$\Rightarrow \alpha = \beta - 0$ $\quad (\because a_n \to 0$ as $n \to \infty)$

$\Rightarrow \alpha = \beta$

$\therefore (S_{2n})$ & (S_{2n-1}) both Converge to the Same Limit l

where $l = \alpha = \beta$.

$\therefore (S_n) \to l$ $\quad (\because n$ is either even or odd$)$

hence $\sum (-1)^{n+1} a_n$ converges to l

$\because (S_n)$ is the SOPS of the given series.

Exp : Show that the Series

$$\sum (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\cdots$$

is a cgt Series.

sol^n: here $a_n = \frac{1}{n}$

Since $n+1 > n \ \forall n$

$\Rightarrow \frac{1}{n} > \frac{1}{n+1} \ \forall n$

$\Rightarrow a_n > a_{n+1} \ \forall n$

$\therefore$ the terms of the given Series are decreasing

Also, $a_n = \frac{1}{n} \to 0$ as $n \to \infty$ $\quad \therefore a_n \to 0$ as $n \to \infty$

hence, by Leibniz test

the given Series is a cgt series.

(Please Note; $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is a divergent Series