## Ratio test for an infinite series

Thm: Let $\sum_{n=1}^{\infty} a_n$ be a +ve term series such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

Then if $\begin{cases} l<1, & \sum a_n \text{ is a cgt series.} \\ l>1, & \sum a_n \text{ will be a div. series} \\ l=1, & \sum a_n \text{ may converge or diverge} \end{cases}$

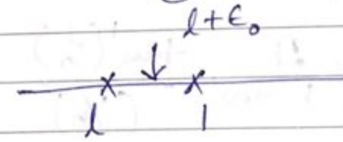
i.e. the test fails for $l=1$

Pf: given $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

$\therefore$ for $\epsilon>0$, $\exists\, m \in \mathbb{N}$ : $\left|\frac{a_{n+1}}{a_n} - l\right| < \epsilon \quad \forall\, n \geq m$

$$\Rightarrow \quad \underbrace{l-\epsilon < \frac{a_{n+1}}{a_n} < l+\epsilon \quad \forall\, n\geq m}_{(1)}$$

Case I : when $l<1$




Choose $\epsilon = \epsilon_0$ : $l+\epsilon_0 < 1$

Let $l+\epsilon_0 = r$, then $r<1$ ($r>0$ as well $\because l \geq 0$ as $\sum a_n$ is a +ve term series)

by (1) $\frac{a_{n+1}}{a_n} < l+\epsilon_0$ (as well) $= r \quad \forall\, n\geq m$

$$\Rightarrow \frac{a_{n+1}}{a_n} < r \quad \forall\, n\geq m$$

$\frac{a_{n+1}}{a_n} < r \quad \forall\, n \geq m$

Put $n = m, m+1, m+2, \cdots, (n-1)$

$\overset{\times}{m} \quad \overset{\times}{n}$

we get

$$\left.\begin{array}{l} \frac{a_{m+1}}{a_m} < r \\ \frac{a_{m+2}}{a_{m+1}} < r \\ \text{-----} \\ \frac{a_n}{a_{n-1}} < r \end{array}\right\}$$

Multiply,

$$\frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdots \frac{a_n}{a_{n-1}} < r^{n-m}$$

$\therefore$ we have

$$\frac{a_n}{a_m} < \frac{r^n}{r^m}$$

$$\Rightarrow a_n < \left(\frac{a_m}{r^m}\right) \cdot r^n$$

which is of the form $a_n < K b_n$

where $K > 0$

Since $\sum r^n$ is a Cgt series (being Geometric Series with $r < 1$)

$\therefore$ by Comparison test, $\sum a_n$ is also a Cgt Series

<u>Case 2</u> : when $l > 1$

$\overset{\times}{1} \quad \overset{l-\epsilon_0}{\downarrow} \quad \overset{\times}{l}$

Choose $\epsilon = \epsilon_0$ : $l - \epsilon_0 > 1$

Let $R = l - \epsilon_0$, then $R > 1$

by (1) $l - \epsilon_0 < \frac{a_{n+1}}{a_n} \quad \forall\, n \geq m$ i.e. $R < \frac{a_{n+1}}{a_n} \quad \forall\, n \geq m$

$R < \frac{a_{n+1}}{a_n} \quad \forall\ n \geq m$

Put $n = m, m+1, m+2, \ldots, (n-1)$

$m \quad m+1 \ldots n$

we get

$$\left.\begin{aligned} R &< \frac{a_{m+1}}{a_m} \\ R &< \frac{a_{m+2}}{a_{m+1}} \\ &\cdots \\ R &< \frac{a_n}{a_{n-1}} \end{aligned}\right\} \text{multiply}$$

$$R^{n-m} < \frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdots \frac{a_n}{a_{n-1}}$$

we have $\frac{R^n}{R^m} < \frac{a_n}{a_m}$

$\Rightarrow \left(\frac{a_m}{R^m}\right) R^n < a_n$ which is of the form $k\, b_n < a_n$

Since, $\sum R^n$ is a div. series (geometric series with ~~R~~ $R > 1$)

$\therefore$ by Comparison test $\sum a_n$ will also be a div. series.

→ For $l = 1$, test fails

Exp. (of a cgt series with $l = 1$)

Let $\sum a_n = \sum \frac{1}{n^2}$, here $a_n = \frac{1}{n^2} \Rightarrow a_{n+1} = \frac{1}{(n+1)^2}$

then $\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

i.e. $\underline{l = 1}$

but $\sum a_n = \sum \frac{1}{n^2}$ is a cgt series.

Exp (of a div. series with $l=1$)

Let $\sum a_n = \sum \frac{1}{n}$

here $a_n = \frac{1}{n} \Rightarrow a_{n+1} = \frac{1}{n+1}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{n}{n+1}$ & $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ i.e. $l=1$

but $\sum \frac{1}{n}$ is a div. series.

Thm: Root test (for an infinite series) (Cauchy's nth root test)

Let $\sum a_n$ be a +ve term series such that

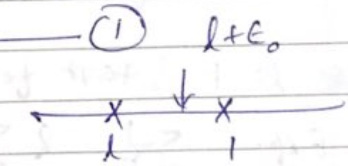
$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = l$$

then if $\begin{cases} l<1, & \sum a_n \text{ will be a cgt series} \\ l>1, & \sum a_n \text{ will be a div. series} \\ l=1, & \sum a_n \text{ may converge or diverge.} \end{cases}$

Pf: given $\lim\limits_{n\to\infty} a_n^{1/n} = l$

$\therefore$ for $\epsilon>0$, $\exists\, m \in \mathbb{N}$ : $\left|a_n^{\frac{1}{n}} - l\right| < \epsilon \quad \forall\, n \geq m$

$$\Rightarrow \quad l-\epsilon < a_n^{\frac{1}{n}} < l+\epsilon \quad \forall\, n \geq m \qquad \text{—— (1)}$$




Case I : when $l<1$

Choose $\epsilon = \epsilon_0$ : $l+\epsilon_0 < 1$

Let $k = l+\epsilon_0$, then $k<1$

by (1) $a_n^{\frac{1}{n}} < l+\epsilon_0$ (as well) $\forall\, n \geq m$

$$a_n^{\frac{1}{n}} < l+\epsilon_0 = r \quad \left(\text{where } 0<r<1\right)$$
$$\forall\, n \geq m$$

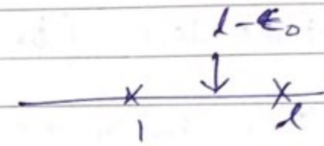
$$\Rightarrow a_n^{\frac{1}{n}} < r \quad \forall\, n \geq m$$

$$\Rightarrow a_n < r^n \quad \forall\, n \geq m$$

Since $\sum r^n$ is a cgt series (geo. series with $r<1$)

$\therefore$ by Comparison test, $\sum a_n$ is also a cgt series

Case II : If $l>1$




Choose $\epsilon=\epsilon_0$ : $l-\epsilon_0>1$

Let $R=l-\epsilon_0$, then $R>1$

by (1) $\quad l-\epsilon_0 < a_n^{\frac{1}{n}} \quad \forall\, n \geq m$

$$\Rightarrow R < a_n^{1/n} \quad \forall\, n \geq m$$

$$\Rightarrow R^n < a_n \quad \forall\, n \geq m$$

Since $\sum R^n$ is a div. series (geo. series with $R>1$)

$\therefore$ by Comparison test, $\sum a_n$ will also be a div. series.

For $l=1$, test fails

Exp. $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$

Hint: $\left( n^{\frac{1}{n}} \to 1 \text{ as } n \to \infty \right)$

(Do it !)

Ex ① Test for the Convergence of the series

$$\sum a_n = \frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots$$

Sol$^n$: here $a_n = \dfrac{(n+1)!}{3^n}$

then $a_{n+1} = \dfrac{(n+2)!}{3^{n+1}}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+2)!}{3^{n+1}} \times \frac{3^n}{(n+1)!} = \frac{(n+2)}{3}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ $\therefore$ the given series is a div. series by Ratio test.

Ex ② Test for Convergence

$$\sum \frac{n!}{n^n}$$

Sol$^n$: here $a_n = \dfrac{n!}{n^n}$

$$\Rightarrow a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{(n+1)!}{n!} \times \frac{n^n}{(n+1)^{n+1}}$$

$$= (n+1) \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{(n+1)}$$

$$= \frac{n^n}{(n+1)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Now

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

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$\therefore$ by Ratio test, $\sum a_n = \sum \frac{n!}{n^n}$ is a cgt series

Ex: Test the series for Convergence

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \cdots$ for $x > 0$

Sol$^n$: here $a_n = \frac{x^{2n-2}}{(2n-2)}$ $\forall\ n > 1$

$\therefore\ a_{n+1} = \frac{x^{2n}}{2n}$

Now, $\frac{a_{n+1}}{a_n} = \frac{x^{2n}}{2n} \cdot \frac{(2n-2)}{x^{2n-2}}$

$= x^2 \cdot \left(\frac{2n-2}{2n}\right)$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2n-2}{2n}\right) \cdot x^2 = x^2$

$\therefore\ l = x^2$

we have, if $l < 1$ i.e. if $x^2 < 1$, then the given Series is cgt

i.e. for $|x| < 1$, the given series will be cgt

& for $l > 1$ i.e. for $|x| > 1$, the given series will be a div. series.

i.e. for $0 < x < 1$, the given series is cgt ($\because x > 0$ given)

& for $x > 1$, " " " " div.

$\longrightarrow$ For $x = 1$, test fails, but for $x = 1$, the series is $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ which we know is div.

for $x=1$

( T.S.

$\sum a_n = 1+\frac{1}{2}+\frac{1}{4}+\cdots$ is a div. series.

let $b_n = \frac{1}{n}$, (here $a_n = \frac{1}{2n-2}$ $n>1$)

then $\frac{a_n}{b_n} = \frac{1}{(2n-2)} \times n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n-2} = \lim_{n\to\infty} \frac{1}{(2-2/n)} = \frac{1}{2}$$

$\neq 0$ & is finite

∴ by Limit Comparison test

then given series for $x=1$ is a div. series

∵ $\sum b_n = \sum \frac{1}{n}$ is a div. series )

Ans.

( ∴ the given series is cgt for $0<x<1$

& is div. for $x \geq 1$ )

→ Application of Root test

Ex. Test for convergence

$$\sum a_n = \sum \left(n^{\frac{1}{n}} - 1\right)^n$$

Sol$^n$. here $a_n = \left(n^{\frac{1}{n}} - 1\right)^n$

$\Rightarrow a_n^{\frac{1}{n}} = n^{\frac{1}{n}} - 1$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(n^{\frac{1}{n}} - 1\right) = 0 \quad (\because n^{\frac{1}{n}} \to 1)$$

∴ $l = 0 < 1$

∴ by Root test, the given series is a cgt series.