Cauchy Criterion for Convergence of Series

Thm: A necessary and sufficient condition for an infinite series $\sum_{n=1}^{\infty} a_n$ to converge is that for each $\epsilon > 0$

$$\exists\ H \in \mathbb{N} : \left| a_{m+1} + a_{m+2} + \cdots + a_n \right| < \epsilon \quad \forall\ n, m \geq H$$

Pf: We know that a series $\sum_{n=1}^{\infty} a_n$ converges

iff SOPS, (S_n) converges where $S_n = a_1 + a_2 + \cdots + a_n$

Now (S_n) converges iff (S_n) is a Cauchy seq.
(by Cauchy convergence criterion for sequences)

i.e. for $\epsilon > 0$, $\exists\ H \in \mathbb{N}$: $|S_n - S_m| < \epsilon \quad \forall\ n, m \geq H$

$$\Leftrightarrow \left| (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_m) \right| < \epsilon$$

(where $n > m$)

$$\Leftrightarrow \left| a_{m+1} + a_{m+2} + \cdots + a_n \right| < \epsilon \quad \forall\ n, m \geq H$$

Exp. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series. (though it appears to be a cgt series)

Sol^n: (S_n) = SOPS of this series is
$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
which we know is a div. seq. (Page (6))

$\therefore$ the given series $\sum \frac{1}{n}$ is a DIVERGENT series.

Positive term Series :

$\sum_{n=1}^{\infty} a_n$ is said to be a series of +ve terms if $a_n > 0 \ \forall n$.

Thm: A series $\sum_{n=1}^{\infty} a_n$ of +ve term is cgt iff $\exists$ a real no. $K > 0$: $a_1 + a_2 + \cdots + a_n < K \ \forall n$

Pf: Let the given series be cgt

$\Rightarrow (S_n) =$ SOPS is cgt seq.

$\Rightarrow (S_n) = (a_1 + a_2 + \cdots + a_n)$ is cgt seq.

$\Rightarrow (S_n)$ is a bdd seq. ($\because$ a cgt seq is bdd)

$\Rightarrow a_1 + a_2 + \cdots + a_n < K$ where K is the bound of the seq. (S_n).

Conversely, let $a_1 + a_2 + \cdots + a_n < K$

$\Rightarrow S_n < K$ — ①

$\Rightarrow (S_n)$ is a bdd seq.

Now $a_n = S_n - S_{n-1}$

Since $a_n > 0 \ \forall n \Rightarrow S_n - S_{n-1} > 0 \ \forall n$

$\Rightarrow S_n > S_{n-1} \ \forall n$

$\Rightarrow (S_n)$ is an inc. seq. — ②

by ① & ②

(S_n) is $\uparrow$ + bdd $\Rightarrow (S_n)$ is cgt seq.

$\Rightarrow \sum a_n$ is cgt series.