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23rd April 2020

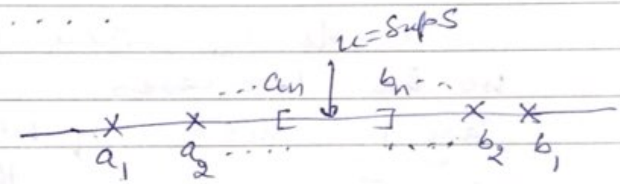
Nested intervals Property

Thm: If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed & bdd intervals, then $\bigcap_{n\in\mathbb{N}} I_n$ is non-empty

(i.e. $\exists$ at least one element x which belongs to I_n $\forall n$)

Pf: we have, $I_n = [a_n, b_n]$; closed & bdd interval $\forall n$

$\therefore I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$



Since $I_1 = [a_1, b_1]$ is the biggest interval containing all the intervals I_n, $n \in \mathbb{N}$

$\therefore I_n \subseteq I_1 \quad \forall n$

$\Rightarrow [a_n, b_n] \subseteq [a_1, b_1] \quad \forall n$

$\Rightarrow \quad \therefore a_1 \le a_n \le b_n \le b_1 \quad \forall n$

$\Rightarrow a_n \le b_1 \quad \forall n$

$\Rightarrow \{a_n : n \in \mathbb{N}\} = (S)$ (say) is a bdd above set & is bdd above by b_1 — ①

by completeness prop., S, must have a Sup.

Let $u = \operatorname{Sup} S$, then $a_n \le u \quad \forall n$ — ①

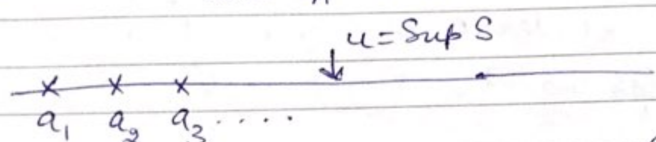
Now, we will show that $u \le b_n \quad \forall n$ — ⊛, If we could show this, then

then by ① $a_n \le u \le b_n \ \forall n$

& we will get that $u \in [a_n, b_n] \ \forall n$

i.e. $u \in \bigcap_{n \in \mathbb{N}} I_n$

Now, T.S. (*) $u \le b_n \ \forall n$

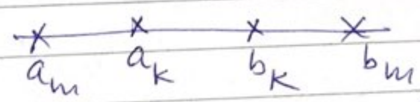


where
$S = \{a_1, a_2, \dots\} = \{a_k : k \in \mathbb{N}\}$ —— Ⓐ

consider b_m, where m is any fixed natural no.
we have two cases

Case I : $m \le k$
Case II : $m > k$
$\Big\}$ we will show that in both the cases, $a_k \le b_m \ \forall k$
where k is as in Ⓐ
(So that, b_m would be an upper bound of S)

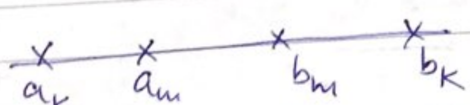
→ For Case I, where $m \le k$
we have $I_m \supseteq I_k$ i.e. $[a_k, b_k] \subseteq [a_m, b_m]$



∴ we get, $a_k \le b_m \ \forall k$ —— ②

→ For Case II, where $m > k$, we have
$I_m \subseteq I_k$

$\Rightarrow [a_m, b_m] \subseteq [a_k, b_k]$



, here also $a_k \le b_m \ \forall k$ —— ③

by ② & ③ $\quad a_k \le b_m \;\forall\, K$

whether $m \le K$ or $m > K$

$\therefore$ in any case b_m becomes an upper bound of the set $S = \{a_K : K \in \mathbb{N}\}$

$\therefore \; \sup S \le b_m \quad (\because \sup S = \text{l.u.b. of } S)$

i.e. $u \le b_m$

Since m is arbitrary, $\therefore u \le b_n \;\forall\, n$

hence ⊛ is proved.

<u>Note</u>: $u = \sup\{a_1, a_2, \dots\} \in \bigcap_{n \in \mathbb{N}} I_n$

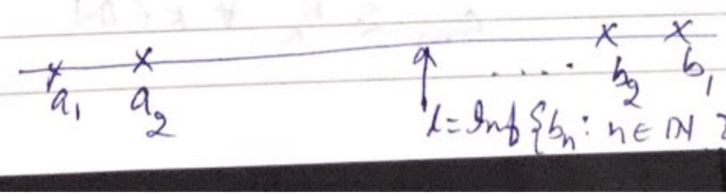
$\Rightarrow$ ~~I_n~~ In the next thm (Q.10)
we will show that

$\inf\{b_1, b_2, \dots\}$ also belongs to $I_n \;\forall\, n$

i.e. $l = \inf\{b_1, b_2, \dots\} \in \bigcap_{n \in \mathbb{N}} I_n$ as well.

<u>Q.10</u> / <u>Ex 2.5</u> If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed & bdd. intervals, then show that $l = \inf\{b_n : n \in \mathbb{N}\}$ belongs to $\bigcap_{n \in \mathbb{N}} I_n$.

<u>Sol^n</u>: We have $I_n = [a_n, b_n]$ be nested seq. of closed & bdd intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$



Since $I_n \subseteq I_1 \; \forall n$

$\Rightarrow [a_n, b_n] \subseteq [a_1, b_1] \; \forall n$

$\Rightarrow a_1 \leq a_n \leq b_n \leq b_1$

$\Rightarrow a_1 \leq b_n \; \forall n$

$\Rightarrow a_1$ is a lower bound of the set

$\{b_K : K \in \mathbb{N}\} = B$ (Say)

Let $l = \cancel{s}$ $\text{Inf } B$, then $l \leq b_K \; \forall K \in \mathbb{N}$ — (1)

Now, we will show that

$a_K \leq l \;\; \forall K \in \mathbb{N}$ — (*)

then we will get

$a_K \leq l \leq b_K \;\; \forall K \in \mathbb{N}$

and ~~ie.~~ that way

we will get that l belongs to $[a_K, b_K]$ $\forall K \in \mathbb{N}$

ie., then l will belong to $\bigcap_{K \in \mathbb{N}} [a_K, b_K]$

and we are done.

Now, T.S. (*) is true ie. T.S. $a_K \leq l \; \forall K \in \mathbb{N}$

we have $B = \{b_K : K \in \mathbb{N}\}$

Consider a_m, where m is a fixed natural no.

we have two cases;

Case I : $m \leq K$; we will show that, in both the cases $a_K \leq l$.

Case II : $m > K$

T.S. this we will prove that

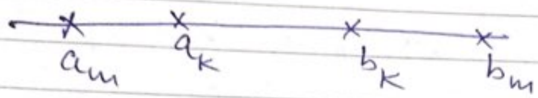
$a_m \leq b_K \;\; \forall K \in \mathbb{N}$

we have

For Case I : $m \leq K \quad \forall K$

$\Rightarrow I_m \supseteq I_K$

i.e. $[a_K, b_K] \subseteq [a_m, b_m]$



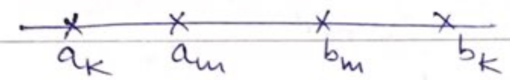
as we see, $a_m \leq b_K \quad \forall K$ —— (1)

For Case II : $m > K \quad \forall K$

$\Rightarrow I_m \subseteq I_K$

$\therefore$ we have

$[a_m, b_m] \subseteq [a_K, b_K]$



here also $a_m \leq b_K \quad \forall K$ — (2)

$\therefore$ by (1) & (2) $a_m \leq b_K \quad \forall K$ (m is fixed & K is vari

$\Rightarrow a_m$ is a lower bound

bu for the set $\{b_K : K \in \mathbb{N}$

$= B$

Since l is the g.l.b. of B

$\therefore a_m \leq l$ for this fixed m

but m is arbitrary, $\therefore a_n \leq l \quad \forall n$

$\therefore$ (*) is proved.

Last thm of this section

Thm 2.5.3 If $I_n = [a_n, b_n] : n \in \mathbb{N}$ be a nested sequence of closed & bdd intervals such that

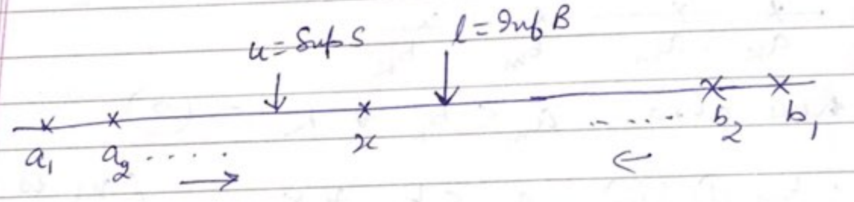
$b_n - a_n$ = length of the interval I_n

satisfy $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$

then $l = u \in \bigcap_{n \in \mathbb{N}} I_n$ is unique,

i.e. $l = u$ is the only pt. which belongs to $\bigcap_{n \in \mathbb{N}} I_n$.

Pf:



where $S = \{a_n : n \in \mathbb{N}\}$

& $B = \{b_k : k \in \mathbb{N}\} = \{b_n : n \in \mathbb{N}\}$

[Actually, $[u, l]$ will belong to $\bigcap_{n \in \mathbb{N}} I_n$

& $\bigcap_{n \in \mathbb{N}} I_n \subseteq [u, l]$

i.e. $[u, l] = \bigcap_{n \in \mathbb{N}} I_n$] — (**)

(we will prove it later!)

for the time being, we are using it

We have

$$\bigcap_{n \in \mathbb{N}} I_n = [u, l]$$

Since in this thm, we are given that

$$\inf \{(b_n - a_n) : n \in \mathbb{N}\} = 0$$

(where length of each $I_n (= b_n - a_n)$)

$\therefore$ we get that for some m

$$0 \leq b_m - a_m < \epsilon \quad \text{—— (A)}$$

as we know that $a_m \leq u$

& $b_m \geq l$

$$\therefore \quad b_m - a_m \geq l - u \geq 0$$

$\therefore$ we get $0 \leq l - u \leq b_m - a_m < \epsilon$

by (A)

i.e. $0 \leq l - u < \epsilon$ for every $\epsilon > 0$

$$\therefore \; l - u = 0 \Rightarrow l = u$$

$\therefore$ $l = u$ is the only pt which belongs to $\bigcap_{n \in \mathbb{N}} I_n$ if the lengths of the intervals are becoming zero.

Now To prove (**)

i.e. $[u, l] = \bigcap_{n \in \mathbb{N}} I_n$

Let $x \in [u, l]$

$\Rightarrow u \leq x \leq l$

Since $l = \inf\{b_n : n \in \mathbb{N}\} \Rightarrow l \leq b_n \;\forall n$

Also $u = \sup\{a_n : n \in \mathbb{N}\} \Rightarrow u \geq a_n \;\forall n$ $\Big\} \Rightarrow$

$\therefore$ we get

$u \le x \le l$

also $l \le b_n \;\forall n$

& $u \ge a_n \;\forall n$

i.e. $a_n \le u \le x \le l \le b_n \;\forall n$

$\Rightarrow a_n \le x \le b_n \;\forall n$

$\Rightarrow x \in \cap [a_n, b_n] \;\forall n$

$\Rightarrow x \in I_n \;\forall n$ i.e. $x \in \cap I_n$

$\Rightarrow [u, l] \subseteq \bigcap_{n \in \mathbb{N}} I_n$ —— (1)

Now, T.S. $\cap I_n \subseteq [u, l]$

Let $x \in \cap I_n \Rightarrow x \in I_n \;\forall n$

$\Rightarrow a_n \le x \le b_n \;\forall n$

$\Rightarrow x$ is a l.b. of $B = \{b_n\}$

$\Rightarrow x \le l = \inf B$ —— (2)

$\Rightarrow x$ is an u.b. of $S = \{a_n : n \in \mathbb{N}\}$

$\therefore \sup S \le x$

i.e. $u \le x$ —— (3)

by (2) & (3) $u \le x \le l \Rightarrow x \in [u, l]$

$\Rightarrow \cap I_n \subseteq [u, l]$ —— (4)

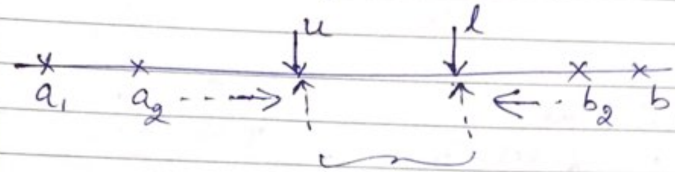
by (1) & (4) $[u, l] = \bigcap_{n \in \mathbb{N}} I_n$

Remark: For nested intervals I_n's

Note: In general $\bigcap_{n \in \mathbb{N}} I_n = [u, l]$

where $u = \sup\{a_n : n \in \mathbb{N}\}$

& $l = \inf\{b_n : n \in \mathbb{N}\}$



$= \bigcap_{n \in \mathbb{N}} I_n$

Exp:

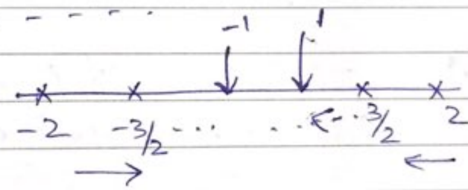
Let $I_n = \left[-1-\frac{1}{n}, \ 1+\frac{1}{n}\right]$

here $I_1 = [-2, 2]$

$I_2 = \left[-\frac{3}{2}, \frac{3}{2}\right]$

$I_3 = \left[-1-\frac{1}{3}, \ 1+\frac{1}{3}\right] = \left[-\frac{4}{3}, \frac{4}{3}\right]$

etc. - - - - -



$\bigcap_{n \in \mathbb{N}} I_n = [-1, 1]$ (take $n \to \infty$ in I_n)

Note: $-1 = \sup\left\{-1-\frac{1}{n} : n \in \mathbb{N}\right\}$

& $1 = \inf\left\{1+\frac{1}{n} : n \in \mathbb{N}\right\}$

—— o ——

(Section 2.5 (Bartle) is over), [Shall be doing the last topic (Section 11 of Chapter in my next lecture!]