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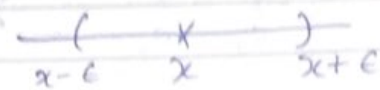
Chapter 11 (Bartle) Open & closed sets in $\mathbb{R}$

(Section 11.1 to 11.1.3)

Def^n: ϵ-neighborhood of a pt x in $\mathbb{R}$



An open interval around x of length ϵ on each side of x is called an ϵ-neighborhood of x

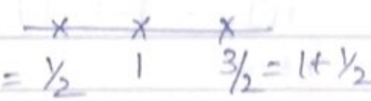
(i.e. $(x-\epsilon, x+\epsilon)$ is called an ϵ-nbhd of x).

→ A general nbhd of a pt x in $\mathbb{R}$ is defined to be a set V (which may or may not be an interval)

Such that V contains an ϵ-nbhd of x

(i.e. $\exists\ \epsilon > 0 : (x-\epsilon, x+\epsilon) \subseteq V$) then V is called a nbhd of x.

Exp:



$\frac{1}{2}$-nbhd of 1 $= \left(\frac{1}{2}, \frac{3}{2}\right)$ is an ϵ-nbhd of 1 where $\epsilon = \frac{1}{2}$

For any general nbhd of 1, we may take

$$V = [0,2] \cup [3,4]$$

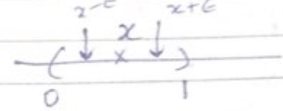
which is also a nbhd of 1 but is not an interval.

Def (open set in $\mathbb{R}$)

A set $G \subseteq \mathbb{R}$ is said to be an open set in $\mathbb{R}$ if it is a nbhd of each of its pts.

i.e. for any $x \in G$, $\exists\ \epsilon > 0 : (x-\epsilon, x+\epsilon) \subseteq G$

Exp ① Show that the set $G = (0,1)$ is an open set.



Let $x \in (0,1) = G$

then $0 < x < 1$

we have to take an $\epsilon > 0 : \underbrace{x+\epsilon < 1 \ \& \ x-\epsilon > 0}$ ①

i.e. $(x-\epsilon, x+\epsilon)$ should be contained in $(0,1)$

by ①; ϵ should be: $\epsilon < 1-x \ \& \ \epsilon < x$

$\therefore$ Choose $\epsilon = \min(x, 1-x)$

then $(x-\epsilon, x+\epsilon)$ will be contained in $(0,1)$

Since $\exists$ an $\epsilon > 0 : (x-\epsilon, x+\epsilon) \subseteq (0,1) = G$

$\therefore$ G is a nbhd of each of its pts.

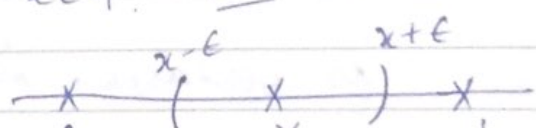
$\Rightarrow$ G is an open set.

Exp ②: Any open interval (a,b) is an open set

Sol^n: Let $G = (a,b)$ T.S. G is an open set

we will show that G is a nbhd of each of its pts.

Let $x \in G$ T.S. $\exists\ \epsilon > 0 : (x-\epsilon, x+\epsilon) \subset G = (a,b)$



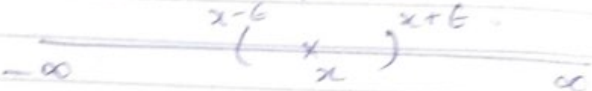
To get such an $\epsilon > 0$,

we choose $\epsilon > 0$ such that

$x+\epsilon < b$ & $x-\epsilon > a$ then $(x-\epsilon, x+\epsilon)$ will be contained in $(a,b) = G$

$\therefore$ G is a nbhd of each of its pts & hence G is an open set.

Exp. (3) $\mathbb{R} = (-\infty, \infty)$ is an open set.



T.S. $\mathbb{R}$ is open, we have to show that $\exists\ \epsilon > 0 : (x-\epsilon, x+\epsilon) \subseteq \mathbb{R}$ for $x \in \mathbb{R}$

here we can take any $\epsilon > 0$

let's take $\epsilon = 1$, then $(x-1, x+1) \subseteq \mathbb{R}$

$\therefore \mathbb{R}$ is a nbhd of $x \in \mathbb{R}$

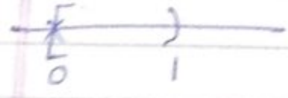
$\Rightarrow \mathbb{R}$ is an open set.

Def^n: (for Not an open set)

A set G is not an open set if there exists a pt in G for which G is not a nbhd of that pt

Exp. $G = [0,1)$, here G is a nbhd of all of its pts in $(0,1)$

but G is not a nbhd of 0



[$\because \nexists$ any $\epsilon > 0$ such that $(0-\epsilon, 0+\epsilon)$ is contained in $G = [0,1)$]

$\therefore$ G is not an open set.

Note: Not an open set doesn't mean that it is a closed set

Defⁿ (Closed Set)

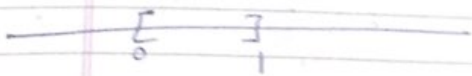
A set $F \subseteq \mathbb{R}$ is said to be a closed set if the complement of F in $\mathbb{R}$ is an open set in $\mathbb{R}$.

Exp. ① $F = [0,1]$ is a closed set

$\because$ its complement in $\mathbb{R} = \mathbb{R} \setminus F$

$= (-\infty, 0) \cup (1, \infty)$ is an open set

(as $(-\infty, 0)$ & $(1, \infty)$ both are open sets)



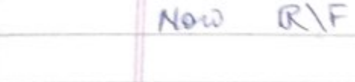
Exp ② : Any closed interval is a closed set

Solⁿ: Let $F = [a, b]$ T.S. F is a closed set

we will show that $\mathbb{R} \setminus F$ is an open set,

Now $\mathbb{R} \setminus F = (-\infty, a) \cup (b, \infty)$

as we know that $(-\infty, a)$ & (b, ∞) are open sets (being nbhd of each of its pts)



$\therefore \mathbb{R} \setminus F$ is an open set

$\Rightarrow F$ is a closed set.

Exp. → The empty set ϕ is an open set in $\mathbb{R}$

(By convention we assume ϕ (empty set) is an open set

as $\nexists$ any pt in the empty set ϕ for which this set is not a nbhd of that pt

$\therefore$ we take ϕ to be an open set)

Exp : Φ (empty set) is a closed set.

Sol^n: By def^n of closed set, A set is closed if its complement is open set.

Now, the complement of Φ (empty set) in $\mathbb{R} = \mathbb{R} \setminus \Phi = \mathbb{R}$ & we know that $\mathbb{R}$ is an open set $\therefore$ Φ is a closed set

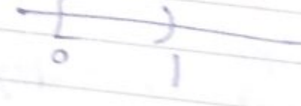
Note (1): A set may be both open & closed at the same time

exp. $\mathbb{R}$ & Φ ($\mathbb{R}$ is a closed set as well $\because \mathbb{R} \setminus \mathbb{R} = \Phi$ is also an open set

Note → (2) A set may be neither closed nor open at the same time.

Exp. $G = [0, 1)$, it is not an open set $\because$ it is not a nbhd of 0

→ it is not a closed set either $\because$ its complement in $\mathbb{R} = \mathbb{R} \setminus G$
$= (-\infty, 0) \cup [1, \infty)$
is not an open set
$\because$ its complement is not a nbhd of 1



$\therefore$ Complement is not open
$\therefore$ G is not a closed set.

Result: ①

Arbitrary union of open sets is open.

Pf:

Let A_i is open set $\forall i$

Then T.S. $\bigcup_i A_i$ is an open set

$\because$ if $x \in \bigcup_i A_i \Rightarrow x \in A_K$ for some K

& Since A_K is openset

$\therefore \exists\ (x-\epsilon, x+\epsilon) \subseteq A_K \subseteq \bigcup_i A_i$

$\Rightarrow \exists\ \epsilon > 0 : \ (x-\epsilon, x+\epsilon) \subseteq \bigcup_i A_i$

$\Rightarrow \bigcup_i A_i$ is a nbd of each of its pts

$\therefore \bigcup_i A_i$ is an open set.

Q.4 Ex 11.1

Arbitrary intersection of open sets may not be an open set.

Let $I_n = (0, 1+\frac{1}{n})$ $n \in \mathbb{N}$, (here I_n is an open set for each $n \in \mathbb{N}$)

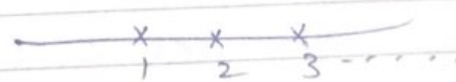
But Then $\bigcap_{n=1}^{\infty} (0, 1+\frac{1}{n})$

$= (0, 1]$ which is not an open set as it is not a nbd of 1

Q.5 (Ex 11.1) The set of natural nos. $\mathbb{N}$ is a closed set in $\mathbb{R}$

Sol^n: T.S. $\mathbb{N}$ is a closed set

we have to show that $\mathbb{R} \setminus \mathbb{N}$ is an open set



Now $\mathbb{R} \setminus \mathbb{N}$

$= (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \cdots$

which is an open set, being the

being arbitrary union of open sets

$(-\infty, 1), (1,2), (2,3), \ldots$

$\mathbb{R} | \mathbb{N}$ is open set (Result ①)

$\therefore$ $\mathbb{N}$ is a closed set.

Q.7 (Ex 11.1) Show that the set $\mathbb{Q}$ (Rational nos.) is neither open nor closed.

Solⁿ:

→ $\mathbb{Q}$ is not an open set : let $x \in \mathbb{Q}$

then $\nexists$ any $\epsilon > 0$: $(x-\epsilon, x+\epsilon) \subseteq \mathbb{Q}$

($\mathbb{Q}$ can't contain any interval $\because$ there would be irrational nos. as well)

$\therefore$ $\mathbb{Q}$ is not an open set.

→ $\mathbb{Q}$ is not a closed set either $\because$ its complement in $\mathbb{R}$ = The set of irrational nos.

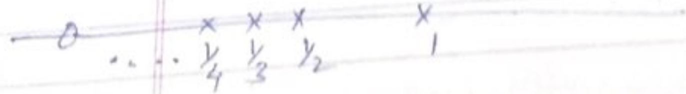
which also is not an open set $\because$ irrational nos. also can't contain any interval.

$\therefore$ $\mathbb{Q}$ is neither closed nor open

Q.6 (Ex 11.1) Show that the set

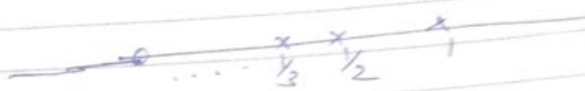
$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

is not a closed set



Solⁿ: If it is a closed set then its complement should

be an open set.
we will show that its Complement is not an open set.
Now, the Complement of A in $\mathbb{R}$
$= \mathbb{R} \setminus A$



$\mathbb{R} \setminus A = (-\infty, 0] \cup S \cup (1, \infty)$

where $S = \{x : 0 < x < 1 \ \& \ x \notin A\}$

but as we see, $\mathbb{R} \setminus A$ is not a nbhd of 0
($\because$ for any $\epsilon > 0$, $(0-\epsilon, 0+\epsilon)$ would Contain pts of A as well by Arch. Prop.)

$\Rightarrow$ $\mathbb{R} \setminus A$ is not an open set
$\therefore$ A is not a closed set.

(ii) Show that $A \cup \{0\}$ is a closed set
Solⁿ: Now
the Complement of $A \cup \{0\}$ in $\mathbb{R}$
is $= (-\infty, 0) \cup S \cup (1, \infty)$
which is an open set.

Q.8 (EX 11.1) If G is an open set & F is a Closed set, Show that $F \setminus G$ is a Closed set
Solⁿ: we will Show that $\mathbb{R} \setminus (F \setminus G)$ is an open set
Now, $\mathbb{R} \setminus (F \setminus G)$
$= (\mathbb{R} \setminus F) \cup (\mathbb{R} \cap G)$ which is open se

($\because$ F closed $\Rightarrow \mathbb{R} \setminus F$ open set
Also, G is open & $\mathbb{R}$ is also open
$\therefore$ $\mathbb{R} \cap G$ is an open set)

$\therefore F \setminus G$ a clos s

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