Lecture (11) Page (66)

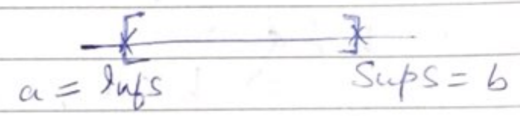
21st April 2020

Ex 2.5 (Bartle)

Q.3 If $S \subseteq \mathbb{R}$ is a non-empty bdd set, & $I_S = [\text{Inf} S, \text{Sup} S]$, Show that (i) $S \subseteq I_S$

(ii) Moreover, If J is any closed bdd interval containing S, show that $I_S \subseteq J$.

Sol^n:



Let $a = \text{Inf} S$
& $b = \text{Sup} S$
given I_S is an interval $(= [a, b])$ (closed interval)

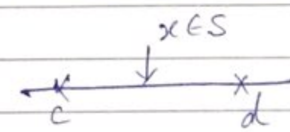
Since $b = \text{Sup} S \Rightarrow x \leq b \ \forall x \in S$
& $a = \text{Inf} S \Rightarrow x \geq a \ \forall x \in S$

$\Rightarrow a \leq x \leq b \ \forall x \in S$
$\Rightarrow S \subseteq [a, b] = I_S$

i.e. $S \subseteq I_S$
$\therefore$ (i) is ~~proved~~ . proved .

To prove (ii) Let $J = [c, d]$ (Say)
(given that J is a closed & bdd interval)

Such that $S \subseteq J$
T.S. $I_S \subseteq J$.



Let $x \in S$, Then $x \in J$ (given)

T.S. $I_S = [\text{Inf} S, \text{Sup} S] = [a, b] \subseteq J = [c, d]$

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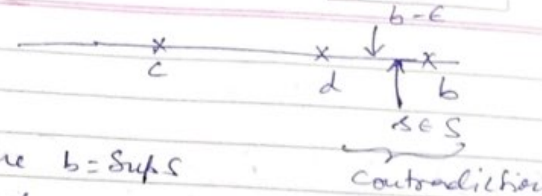


firstly, we will show that

$b \le d$, where $b = \text{Sup } S$

Let if possible $b > d$

if we consider $b-\epsilon$, then $\exists\ s \in S : s > b-\epsilon$ $: b-\epsilon > d$ (Sup thm.)

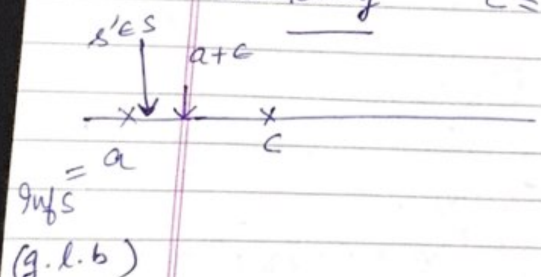
then s will be bigger than d

which is not possible $\because S \subseteq [c, d] = J$

$\therefore b \not> d$

$\Rightarrow b \le d$, — (*)

Similarly $c \le a$ where $a = \text{Inf } S$



Let if possible $c > a$

Consider $a+\epsilon$,

then $\exists\ s' \in S : s' < a+\epsilon < c$ (Inf thm)

then s' will be less than c

but $s' \in S$ $\therefore$ it can't be less than c

as $S \subseteq J = [c, d]$

$\therefore c \not> a$

$\Rightarrow c \le a$ — (**)

by (*) & (**) $c \le \underbrace{a \le b}_{I_S} \le d$

$\Rightarrow I_S \subseteq [c, d] = J$

Nested Intervals

Def^n: Let (I_n) be a sequence of intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

(In general $I_n \supseteq I_{n+1}\ \forall n$)

then (I_n) is called a nested seq. of ~~nested~~ intervals

Exp ① $I_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$, $n \in \mathbb{N}$

~~where~~ then $I_1 = [-1, 1]$ here $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

$I_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$I_3 = \left[-\frac{1}{3}, \frac{1}{3}\right]$

$- - - -$

Then $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$ i.e. 0 is the only pt which belongs to all the intervals $I_n : n \in \mathbb{N}$

Exp ② $I_n = \left[0, \frac{1}{n}\right]$

here $I_1 = [0, 1]$

$I_2 = \left[0, \frac{1}{2}\right]$

$I_3 = \left[0, \frac{1}{3}\right]$

$- - - -$

Since $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

$\therefore (I_n)$ is a nested sequence of intervals.

here also $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$,

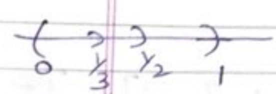
Exp. 3 let $I_n = (0, \frac{1}{n})$ (open intervals)

here $I_1 = (0, 1)$
$I_2 = (0, \frac{1}{2})$
$I_3 = (0, \frac{1}{3})$
- - - - .

all are open intervals.

we still have $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

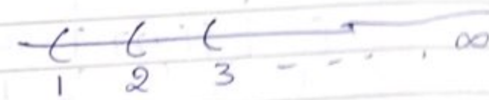
i.e. I_n is a nested seq. of intervals



but $\bigcap_{n \in \mathbb{N}} I_n$ is empty.

Exp. 4 let $I_n = (n, \infty)$

here $I_1 = (1, \infty)$
$I_2 = (2, \infty)$
$I_3 = (3, \infty)$
- - - .



(open rays)

we have $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

$\therefore$ the given seq. of intervals is nested,

but $\bigcap_{n \in \mathbb{N}} I_n$ is empty.

Exp. 5 let $I_n = [n, \infty)$ (closed rays)

then $I_1 = [1, \infty)$
$I_2 = [2, \infty)$
- - - . .

is a seq. of nested intervals

where $\bigcap_{n \in \mathbb{N}} I_n$ is still empty

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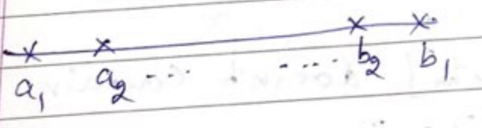
Remark : In Exp. (1) & (2)
the intervals are closed & bdd
but in Exp. (3), we have open (but bdd) intervals.
→ In exp. (4) & (5), the intervals are unbdd

Note : If the intervals are closed & bdd
then their intersection can't be empty
which is called Nested intervals Property
(we will prove this Property later)

Ex 2.5

Q.6. If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a nested sequence of intervals & If $I_n = [a_n, b_n]$
Show that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$
& $b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$

Sol^n: we have $I_n \supseteq I_{n+1}$



$\Rightarrow [a_n, b_n] \supseteq [a_{n+1}$

$\therefore [a_1, b_1] \supseteq [a_2, b_2]$
$[a_2, b_2] \supseteq [a_3, b_3]$ & so on

$\Rightarrow a_1 \leq a_2 \leq a_3 \leq \cdots$

& $\left.\begin{array}{l} b_2 \leq b_1 \\ b_3 \leq b_2 \\ \text{etc.} \end{array}\right\} \Rightarrow b_1 \geq b_2 \geq b_3 \geq \cdots$

Ex 2.5

Q.7 Let $I_n = [0, \frac{1}{n}]$ $\forall n \in \mathbb{N}$. Prove that
$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

Solⁿ: we have
$I_1 = [0, 1]$
$I_2 = [0, \frac{1}{2}]$
$I_3 = [0, \frac{1}{3}]$
$- - - - .$

Since $0 \in I_n \ \forall n \Rightarrow \{0\} \subseteq \bigcap_{n \in \mathbb{N}} I_n$

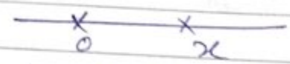
T.S. $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$

We will show that 0 is the only pt which belongs to the intersection of all I_n's

Let $x \neq 0$, $x \in \mathbb{R}$

T.S. $x \notin I_m$ for some $m \in \mathbb{N}$.

Now if $x > 0$, then



Consider $\frac{1}{x}$ which will also be greater than 0
then by Archimedean prop.
$\exists \, m \in \mathbb{N}$: $\frac{1}{x} < m \Rightarrow \frac{1}{m} < x$

then the interval $[0, \frac{1}{m}]$ doesn't contain x
i.e. $x \notin [0, \frac{1}{m}] = I_m$

$\therefore$ x can't belong to the intersection of all I_n's.

$\therefore$ if $x \neq 0$, then $x \notin \bigcap I_n$ but $0 \in \bigcap I_n$ $\Big\} \Rightarrow \bigcap I_n = \{0\}$

(if $x < 0$, then obviously $x \notin$ any interval I_n)

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Ex. 2·5 (Bartle)

Q.8 Let $I_n = (0, \frac{1}{n}) : n \in \mathbb{N}$, Prove that

$$\bigcap_{n \in \mathbb{N}} I_n = \phi$$

Sol^n: here I_n is open interval $\forall n$

T.S $\cap I_n$ is empty

We have $I_1 = (0, 1)$
$I_2 = (0, \frac{1}{2})$
$I_3 = (0, \frac{1}{3})$
$\ldots$

$0 \notin I_n$ for any n. $\Rightarrow 0 \notin \cap I_n$

if $x < 0$ then also, $x \notin I_n$ for any n

if $x > 0$, Then $\exists\, m \in \mathbb{N} : \frac{1}{x} < m$ (Arch. Prop.)

i.e. $\frac{1}{m} < x$

$\Rightarrow x \notin (0, \frac{1}{m}) = I_m$

$\Rightarrow x \notin \bigcap_{n \in \mathbb{N}} I_n$

$\therefore$ we get, no real no. x ($x = 0$, $x > 0$, $x < 0$)

can belong to $\bigcap_{n \in \mathbb{N}} I_n$

$\therefore \bigcap_{n \in \mathbb{N}} I_n$ is empty.

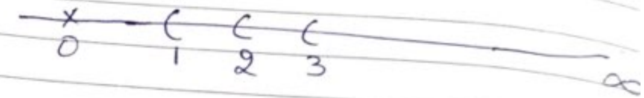
Ex 2.5 (Bartle)

Q.9. Let $I_n = (n, \infty)$ $n \in \mathbb{N}$

Prove that $\bigcap_{n \in \mathbb{N}} I_n = \phi$

Sol^n: here I_n is open & unbdd for each n.

Such that



$I_1 = (1, \infty)$

$I_2 = (2, \infty)$

$I_3 = (3, \infty)$ & $I_1 \supseteq I_2 \supseteq \cdots\cdots$

$- - - \cdot\cdot$

T.S. $\bigcap_{n \in \mathbb{N}} I_n$ is empty. , Let $x \in \mathbb{R}$

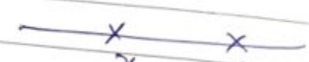
(i) If $x = 0$, then $x \notin I_n$ for any n

(ii) If $x < 0$, then also $x \notin I_n$ for any n

(iii) If $x > 0$

then $\exists\ m \in \mathbb{N}$

$: x < m$



(Arch. Prop.)

(for any real no

$\exists$ a natural no. bigger than x)

then $x \notin (m, \infty) = I_m$

$\therefore$ x can't belong to the intersection

by (i), (ii) & (iii), no real number x can belong to the intersection of I's

$\therefore \bigcap_{n \in \mathbb{N}} I_n = \phi$