17th April 2020

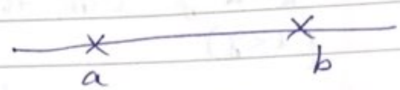
Lecture 10

Section 2.5 (Bartle)

Intervals

(i) Bounded Intervals : Let $a, b \in \mathbb{R}$: $a < b$

→ open Interval : (a, b)



we define $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

(i.e. all the real numbers from 'a' to 'b' excluding a & b.)

→ Closed Interval : $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

i.e. the ends pts a & b are also included

→ Half-open (or half-closed) Intervals

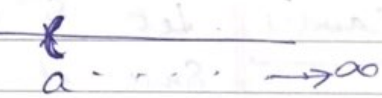
$[a, b)$ or $(a, b]$ → a is not included

↳ a is included but b is not included.

(ii) Unbounded Intervals : (a, ∞), $[a, \infty)$

$(-\infty, b)$, $(-\infty, b]$ or $(-\infty, \infty)$

→ $(a, \infty) = \{x \in \mathbb{R} : a < x\}$



→ $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$

→ $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$

→ $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

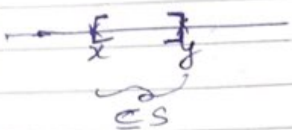
→ $(-\infty, \infty) = \{x \in \mathbb{R}\} = \mathbb{R}$

Characterization Theorem :

Let S be a Subset of $\mathbb{R}$ (which may or may not be an interval)

having at least two pts x & y with the property that the interval $[x, y] \subseteq S$ $(x<y)$

then S itself will be an interval.



Pf: Since S is a non-empty subset of $\mathbb{R}$
There are four possibilities :

Cases

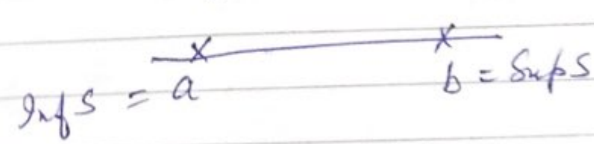
(i) S is a bdd set (bounded above as well as bdd below)

(ii) S is bdd above but not bdd below

(iii) S is bdd below but not bdd above

(iv) S is neither bdd above nor bdd below.

Case(i) Let S be a bdd set
Since it is bdd above $\therefore$ Sup S exists (Completeness prop.)
Let $b = \text{Sup } S$
Also, S is bdd below $\therefore$ Inf S exists
Let $a = \text{Inf } S$



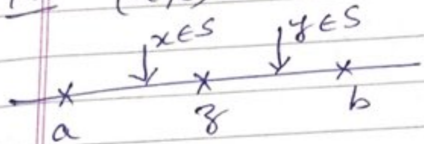
T.S. $S = (a, b)$, we will show that $(a,b) \subseteq S$ & $S \subseteq (a,b)$

Now,

$a \le x \le b \quad \forall x \in S \quad \because b = \text{Sup } S$ & $a = \text{Inf } S$

$\Rightarrow S \subseteq [a, b]$ — ($*$)

T.S. $(a,b) \subseteq S$, Let $z \in (a,b)$, we will show that $z \in S$



Since $z < b = \text{Sup } S = \text{l.u.b. of } S$

$\Rightarrow z$ can't be an u.b. of S

$\therefore \exists\, y \in S :$ $z < y \le b$

Also, $z > a = \text{Inf } S = \text{g.l.b. of } S$

$\Rightarrow z$ can't be a l.b. of S

$\therefore \exists\, x \in S : \quad a \le x < z$

$\therefore$ we get $x \in S$ & $y \in S \Rightarrow [x, y] \subseteq S$ (given condition)

& z lies in $[x, y]$

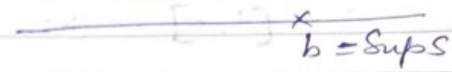
$\therefore z \in S \Rightarrow (a,b) \subseteq S$ — ($**$)

by ($*$) & ($**$) $S = (a, b)$ $\therefore S$ is an interval

<u>Case 2</u> : S is bdd above but not bdd below

Since S is bdd above $\therefore$ $\sup S$ exists
& S is not bdd below $\therefore$ $\inf S$ doesn't exist.

Let $b = \sup S$

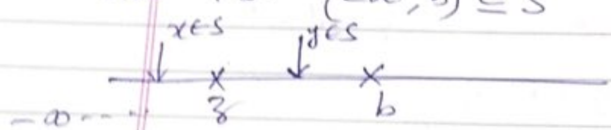


We will show that in this case S will be $(-\infty, b)$.

<u>T.S.</u> $S \subseteq (-\infty, b)$ & $(-\infty, b) \subseteq S$

Since $x \leq b \ \forall \ x \in S$ ($\because b = \sup S$)
$\Rightarrow S \subseteq (-\infty, b]$ — (*)

Now T.S. $(-\infty, b) \subseteq S$, Let $z \in (-\infty, b)$
we will show that $z \in S$



In this case, Since S is not bdd below
$\therefore$ z can't be a lower bound of S

$\therefore \exists \ x \in S : x < z$

Now, $b = \sup S = $ l.u.b. of S

$\therefore$ z can't be an u.b. of S ($\because z < b$)

$\therefore \exists \ y \in S : y > z$, also $y \leq b$

$\therefore z < y \leq b$

So, we get two elements x & y both belonging to S

$\Rightarrow [x, y] \subseteq S$ (given)

$\Rightarrow z \in S \Rightarrow (-\infty, b) \subseteq S$ (**)

$\therefore S = (-\infty, b)$ by (*) & (**)

Case 3 : S is bdd below but not bdd above

Since S is bdd below $\therefore$ Inf S exists

Let $a = \text{Inf } S$

but S is not bdd above $\therefore$ Sup S doesn't exist

Inf S = $a \ldots\ldots \infty$

We will show that in this case, $S = (a, \infty)$

$\therefore$ we will show that $(a,\infty) \subseteq S$

& $S \subseteq (a,\infty)$

Now $a \leq x \quad \forall x \in S \quad (\because a = \text{Inf } S)$

$\Rightarrow S \subseteq (a,\infty)$ — (*)

T.S. $(a,\infty) \subseteq S$

Let $z \in (a,\infty)$ T.S. $z \in S$

$x \in S \quad y \in S$

$a \quad z \quad \ldots \infty$

Since $z > a = \text{Inf } S = \text{g.l.b. of } S$

$\Rightarrow z$ is not a l.b. of S

$\therefore \exists\, x \in S : x < z$.

Now, S is not bdd above

$\therefore z$ can't be an upper bound of S

$\therefore \exists\, y \in S : y > z$

So, we get two elements x & y which belong to S.

by the given hypothesis, $[x, y] \subseteq S$

$\Rightarrow z \in S$ $\because x < z < y$

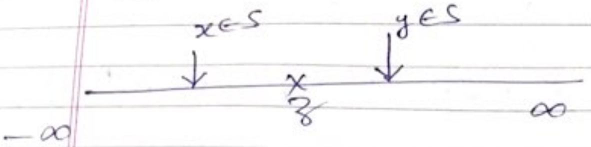
$\Rightarrow (a,\infty) \subseteq S$ — (**)

$\Rightarrow S = (a,\infty)$

by (*) & (**)

Case 4: S is neither bdd above nor bdd below

T.S. that in this case $S = (-\infty, \infty) = \mathbb{R}$



Since $S \subseteq \mathbb{R}$ we will only show that $\mathbb{R} \subseteq S$.

Let $z \in \mathbb{R}$

we will show that $z \in S$

Since z is not a l.b. of S (lower bound doesn't exist)

$\therefore \exists \underline{x \in S} : x < z$

Also, z is not an u.b. of S ($\because$ upper bound doesn't exist)

$\therefore \exists \underline{y \in S} : y > z$

$\therefore$ we get two elements x & y where both belonging to S ($x < z < y$)

by the hypothesis $[x, y] \subseteq S$

$\Rightarrow z \in S$

$\Rightarrow \mathbb{R} \subseteq S \quad \therefore \mathbb{R} = S$

Ex 2.5

(2) If $S \subseteq \mathbb{R}$, show that S is bdd iff $\exists$ a closed and bdd interval $I : S \subseteq I$

Sol^n: Let S be bdd $\Rightarrow$ Sup S & Inf S both exist

Let $\left.\begin{array}{l} a = \text{Inf } S \\ b = \text{Sup } S \end{array}\right\} \Rightarrow a \leq x \leq b \;\; \forall x \in S$

$\Rightarrow S \subseteq [a, b] = I$

then I is a closed & bdd interval.

Conversely: Let $S \subseteq I$ where I is a closed & bdd interval, Let $I = [a', b']$

Since $S \subseteq I \Rightarrow a' \leq x \leq b' \;\; \forall x \in S \quad \therefore S$ is a bdd set.