Limit Comparison test

Let $\sum a_n$ & $\sum b_n$ be two +ve term series (Infinite) such that
$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \quad \text{where } l \neq 0 \text{ \& is finite.}$$
then the two series $\sum a_n$ & $\sum b_n$ will behave alike (i.e. If anyone of them is cgt then other is also cgt & If anyone of them is div. then other is also div.)

Pf: Since $\sum a_n$ & $\sum b_n$ be +ve term series

$$\therefore \frac{a_n}{b_n} > 0 \quad \forall n$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} \geq 0 \quad \Rightarrow l \geq 0$$
but $l \neq 0$
$$\therefore l > 0$$

Choose $\epsilon_0 > 0$: $l - \epsilon_0 > 0$, Let $l - \epsilon_0 = k$, then $k > 0$
(take $\epsilon = \epsilon_0$, to fix)

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = l$ (given)

$\therefore$ for each $\epsilon > 0$, $\exists\, m \in \mathbb{N}$: $\left| \frac{a_n}{b_n} - l \right| < \epsilon \quad \forall n \geq m$

$$\Rightarrow l - \epsilon < \frac{a_n}{b_n} < l + \epsilon \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon_0 < \frac{a_n}{b_n} < l + \epsilon_0 \quad \forall n \geq m \text{ (as well)}$$

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$\Rightarrow (l-\epsilon_0) b_n < a_n < (l+\epsilon_0) b_n$ [$b_n > 0$ $\therefore$ inequality remains unchanged)

$\Rightarrow k b_n < a_n < K b_n$

where $k = l - \epsilon_0$
& $K = l + \epsilon_0$ } & both are +ve real nos.

$\therefore$ by basic Comparison test (1)(b)
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Σa_n & Σb_n will converge or div. together.

Exp. (1) Test for Convergence (of the infinite series)

$$\frac{1}{3\cdot 7} + \frac{1}{4\cdot 9} + \frac{1}{5\cdot 11} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$$

here $a_n = \frac{1}{(n+2)(2n+5)} > 0 \quad \forall n$

for large values of n, $a_n \sim \frac{1}{2n^2}$

$\therefore$ Let $b_n = \frac{1}{n^2}$

then $\frac{a_n}{b_n} = \frac{1}{(n+2)(2n+5)} \times \frac{n^2}{1} = \frac{1}{(1+\frac{2}{n})(2+\frac{5}{n})}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{2}{n})(2+\frac{5}{n})} = \frac{1}{2} = l$$

& $l \neq 0$
& is finite

$\therefore$ Limit Comparison test $\sum \frac{1}{(n+2)(2n+5)}$ & $\sum \frac{1}{n^2}$
will behave alike

Since $\sum \frac{1}{n^2}$ is cgt series [$\because$ p-series with $p>1$ is cgt series]

$\therefore$ the given series is cgt series

Ex: Test for Convergence

$= \frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \cdots$

Sol: here $a_n = \frac{\sqrt{n}}{2n+3} > 0 \;\forall n$

Also $a_n \sim \frac{1}{2\sqrt{n}}$

Let $b_n = \frac{1}{\sqrt{n}}$

then $\frac{a_n}{b_n} = \frac{\sqrt{n}}{(2n+3)} \times \frac{\sqrt{n}}{1} = \frac{n}{2n+3}$

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$ which is finite & non-zero

$\therefore$ the given series $\sum \frac{\sqrt{n}}{2n+3}$

& $\sum \frac{1}{\sqrt{n}}$ will converge or div. together

Since $\sum \frac{1}{\sqrt{n}}$ is a div. series (p-series with $p<1$)

$\therefore$ the given series $\sum \frac{\sqrt{n}}{2n+3}$

will also be a div. series.

Ex: Test for Convergence

$\sum a_n = \sum \left(\sqrt{n^4+1} - n^2\right)$

Sol: here $a_n = \sqrt{n^4+1} - n^2$

On rationalizing $a_n = \frac{\left(\sqrt{n^4+1} - n^2\right)\left(\sqrt{n^4+1} + n^2\right)}{\left(\sqrt{n^4+1} + n^2\right)}$

$= \frac{1}{\sqrt{n^4+1} + n^2} \sim \frac{1}{2n^2}$

Let $b_n = \frac{1}{n^2}$

then $\frac{a_n}{b_n} = \frac{1}{\left(\sqrt{n^4+1} + n^2\right)} \times \frac{n^2}{1}$

$$= \frac{1}{\left(\sqrt{1+\frac{1}{n^4}} + 1\right)}$$

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$ which is finite & non-zero

$\therefore$ the given series & $\sum \frac{1}{n^2}$ will converge or diverge together

Since $\sum \frac{1}{n^2}$ is cgt series $\therefore$

$\therefore \sum\left(\sqrt{n^4+1} - n^2\right)$ is a cgt series.

Exp. Test for Convergence

$$\sum a_n = \frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$$

Sol^n: here $a_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} > 0$

$a_n \sim \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$ (for large values of n)

Let $b_n = \frac{1}{n^{5/2}}$, then $\frac{a_n}{b_n} = \frac{\sqrt{n+1}-1}{(n+2)^3-1} \times n^{5/2}$

$$= \left(\frac{\sqrt{n+1}-1}{\sqrt{n}}\right)\left(\frac{n^3}{(n+2)^3-1}\right)$$

$\therefore$

$$\frac{a_n}{b_n} = \left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right) \cdot \left(\frac{1}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}}\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{1} = 1$$ which is finite & non-zero

$\therefore$ the given series $\sum a_n = \sum \frac{\sqrt{n+1} - 1}{(n+2)^3 - 1}$

& $\sum b_n = \sum \frac{1}{n^{5/2}}$

will Converge or diverge together

Since $\sum \frac{1}{n^{5/2}}$ is a cgt series (p-series with $p>1$)

$\therefore$ the given Series is a cgt series.

Thm: If $\sum a_n$ & $\sum b_n$ be two +ve term series such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then & if $\sum b_n$ is cgt series

$\Rightarrow \sum a_n$ is also a cgt series.

$\frac{a_n}{b_n}$ cgt

Pf:

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$\therefore$ for each $\epsilon > 0$, $\exists$ $m \in \mathbb{N}$:

$$\left|\frac{a_n}{b_n} - 0\right| < \epsilon \quad \forall\ n \geq m$$

$$\Rightarrow \frac{a_n}{b_n} < \epsilon \quad \begin{pmatrix} a_n > 0 \\ \& \ b_n > 0 \end{pmatrix}$$

$\Rightarrow a_n < \epsilon \cdot b_n$ Let $\epsilon = K > 0$

then $a_n < K b_n \quad \forall\, n \geq m$

$\therefore$ by basic Comparison test (1)(a)

if $\sum b_n$ is cgt series, then $\sum a_n$ is also a cgt series.

<u>Ex</u>: If $\sum a_n$ & $\sum b_n$ be two +ve term series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$, then if $\sum b_n$ is div. series then $\sum a_n$ is also div. series.

<u>Pf</u>: Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

$\therefore$ for every large real no. $K > 0$

$\exists\, m \in \mathbb{N}$:

$$\frac{a_n}{b_n} > K \quad \forall\, n \geq m$$

$\Rightarrow \quad a_n > K b_n$

i.e. $\quad K b_n < a_n \quad \forall\, n \geq m$

or $\quad b_n < \frac{1}{K} a_n \quad (K > 0)$

$\therefore$ by basic Comparison test (1)(a)

if $\sum b_n$ is a div. series, then $\sum a_n$ is also div. series.

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Ex

Test for the Convergence, using any test

(i) $\sum \frac{1}{2n+1}$ (Ans: div.) (Do it!)

(ii) $\sum \left(\frac{2^{n+1}}{3^{n-1}}\right) \cdot \frac{1}{n^2}$

Sol^n: here $a_n = \frac{2^{n+1}}{3^{n-1}} \cdot \frac{1}{n^2} = \left(\frac{2}{3}\right)^n \cdot \frac{2}{3^{-1}} \cdot \frac{1}{n^2}$

$= 6\left(\frac{2}{3}\right)^n \cdot \frac{1}{n^2}$

$< \frac{6}{n^2} \quad \forall n$

$\therefore a_n < 6 \cdot \frac{1}{n^2}$ Let $b_n = \frac{1}{n^2}$

then $a_n < K b_n$ where $K = 6 > 0$

$\therefore$ by basic comparison test, $\sum a_n$ is cgt series

($\because \sum b_n = \sum \frac{1}{n^2}$ is a cgt series)

(iii) $\sum \frac{n^2+1}{n^3+1}$ (Do it!)

(Hint: Compare with the harmonic series $\sum \frac{1}{n}$)

(iv) $\sum \left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)^{1/2}$ (Ans: div. series)

as $a_n \not\to 0$ as $n \to \infty$

(nth term test)

→ Ex Show that for $p > 1$
the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ Converges.

(Do it !) use Integral test

→ Check for the Convergence of the Series

$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n}\right)$ using

the following :

(i) The integral test

(ii) The Limit Comparison test

Sol^n : (i) Using the Integral test

Let $f(x) = \tan^{-1}\left(\frac{1}{x}\right) \quad x \geq 1$

Then f is non-neg., ↓ & integrable fn.

∴ by Integral test $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)dx$ will Converge or diverge together

Now $\int_1^{\infty} f(x)dx = \int_1^{\infty} \tan^{-1}\left(\frac{1}{x}\right)dx$,

Integrate by parts, we get $\int_1^{\infty} \tan^{-1}\left(\frac{1}{x}\right)dx$

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$$\int_1^{\infty} \tan^{-1}\left(\frac{1}{x}\right)dx = \tan^{-1}\left(\frac{1}{x}\right)\cdot x\Big]_1^{\infty} - \int_1^{\infty} \frac{(-1/x^2)}{1+\left(\frac{1}{x}\right)^2}\cdot x\,dx$$

$$= \left(0 - \tan^{-1}(1)\right) + \frac{1}{2}\int_1^{\infty} \frac{2x}{1+x^2}\,dx$$

$$= -\frac{\pi}{4} + \frac{1}{2}\log\left(1+x^2\right)\Big]_1^{\infty}$$

$$= -\pi/4 + \infty$$

$$\therefore \int_1^{\infty} \tan^{-1}\left(\frac{1}{x}\right)dx = \infty \qquad (a_n = f(n))$$

$\therefore$ by I. test $\sum a_n$ is a div. series.

(ii) Using Limit Comparison test

Let $b_n = \frac{1}{n}$ where $\sum a_n = \sum \tan^{-1}\left(\frac{1}{n}\right)$ is the given series

then $\frac{a_n}{b_n} = \frac{\tan^{-1}\left(\frac{1}{n}\right)}{(1/n)}$

Now $\lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{(1/n)}$ $\left(\text{we have } \lim_{x\to 0} \frac{\tan^{-1}x}{x} \overset{0/0 \text{ form}}{=} \lim_{x\to 0} \frac{1}{1+x^2} = 1\right)$

$= 1$ which is finite & non-zero.

$\therefore$ by Limit Comparison test

$\sum \tan^{-1}\left(\frac{1}{n}\right)$ & $\sum \frac{1}{n}$ will converge or diverge together

Since $\sum \frac{1}{n}$ is a <u>div. series</u> (p-series with $p=1$)

$\therefore$ the given series is a <u>div.</u> series.