Now, $|z_n - z_m| = |(x_n + y_n) - (x_m + y_m)|$

$$= |(x_n - x_m) + (y_n - y_m)|$$

$$\leq |x_n - x_m| + |y_n - y_m|$$

$$< \epsilon/2 + \epsilon/2 \quad \forall\ n, m \geq \max\{H_1, H_2\}$$

$$= H \text{ (Say)}$$

Then $|z_n - z_m| < \epsilon \quad \forall\ n, m \geq H$

$\Rightarrow (z_n) = (x_n + y_n)$ is a Cauchy seq.

(ii) T.S. $(x_n y_n)$ is a Cauchy seq.

we need to show that $|x_n y_n - x_m y_m| < \epsilon$
for $\epsilon > 0$, $\exists\ H \in \mathbb{N}$: $\forall\ n, m \geq H$

Now $|x_n y_n - x_m y_m| = |x_n(y_n - y_m) + x_n y_m - x_m y_m|$

$$= |x_n (y_n - y_m) + y_m (x_n - x_m)|$$

$$\leq |x_n (y_n - y_m)| + |y_m (x_n - x_m)|$$

$$\leq |x_n|\,|y_n - y_m| + |y_m|\,|x_n - x_m|$$

(here n & m both are variables)

$$\leq K_1 |y_n - y_m| + K_2 |x_n - x_m|$$

(*) where, K_1 & K_2 are the bounds of the sequences (x_n) & (y_n) resp.

$(K_1 > 0$ & $K_2 > 0)$

Since (y_n) is cauchy

$\therefore$ for $\epsilon>0$, $\exists\ H_1 \in \mathbb{N}$: $|y_n - y_m| < \frac{\epsilon}{2K_1}$ $\forall\ n, m \geq H_1$ —(1)

Also, (x_n) is cauchy

$\therefore$ for $\epsilon>0$, $\exists\ H_2 \in \mathbb{N}$: $|x_n - y_m| < \frac{\epsilon}{2K_2}$ $\forall\ n, m \geq H_2$ —(2)

$\therefore$ by (*), (1) & (2)

$$|x_n y_n - x_m y_m| \leq K_1 |y_n - y_m| + K_2 |x_n - x_m|$$

$$< K_1 \cdot \frac{\epsilon}{2K_1} + K_2 \cdot \frac{\epsilon}{2K_2} \quad \forall\ n, m \geq \max\{H_1, H_2\} = H$$

$\Rightarrow |x_n y_n - x_m y_m| < \epsilon \quad \forall\ n, m \geq H$

$\Rightarrow (x_n y_n)$ is a Cauchy seq.

Q.5 If $x_n = \sqrt{n}$, Show that (x_n) satisfies $|x_{n+1} - x_n| \to 0$ as $n \to \infty$ but it is not a Cauchy seq.

Sol^n: $|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = \left|\frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$ as $n \to \infty$

Since $x_n = \sqrt{n}$ is not a bdd seq.

$\therefore$ it is not Cauchy.

Q.6 (Same as Q.5)

Q.7 If (x_n) is a Cauchy seq. such that x_n is an integer for every $n \in \mathbb{N}$, Show that (x_n) is ultimately Const

Sol^n:

Sol^n: (Q.7)

given that (x_n) is a Cauchy seq. where each x_n is an integer.

Now, $|x_n - x_m|$ will be a non-neg. integer ($\because x_n$ is integer $\forall n \in \mathbb{N}$)

Let $K = |x_n - x_m|$, then $K \geq 0$ is a non-neg. integer. for each n & m

Also, (x_n) is Cauchy (here K is a variable dependent on n & m)

$\therefore$ for $\epsilon > 0$, $\exists$ $H \in \mathbb{N}$

: $|x_n - x_m| < \epsilon$ $\forall$ $n, m \geq H$

$\Rightarrow$ $0 \leq K < \epsilon$ $\forall$ $n, m \geq H$ $(*)$

i.e. K is a non-neg. integer less than every +ve ϵ ultimately ($\because (*)$ holds $\forall n, m \geq H$)

$\Rightarrow$ $K = 0$ ultimately

i.e. $|x_n - x_m| = 0$ ultimately

$\Rightarrow$ $x_n = x_m$ ultimately

$\Rightarrow$ (x_n) is a const seq. ultimately.

Q. 8 Show directly that a bdd + $\uparrow$ seq. is a Cauchy seq.

Sol^n: Let (x_n) be a bdd & increasing seq.

T.S. (x_n) is Cauchy

i.e. T.S. for $\epsilon > 0$, $\exists\, H \in \mathbb{N}$:

$$|x_n - x_m| < \epsilon \quad \forall\, n, m \geq H$$

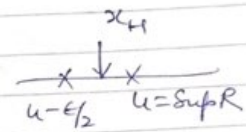
Since (x_n) is bdd

$\therefore \quad R = \{x_n : n \in \mathbb{N}\}$ which is the range set of (x_n) is a bdd set.

Let $u = \sup R$ (it exists by Completeness Prop. of $\mathbb{R}$)

$\Rightarrow \quad x_n \leq u \quad \forall\, n$ — (1)

Consider $u - \epsilon/2$, then $\exists\, x_H \in R$:



$$u - \epsilon/2 < x_H$$

Let $n \geq H$, then $x_n \geq x_H$ ($\because (x_n)$ is $\uparrow$)

$\Rightarrow \quad u - \epsilon/2 < x_H \leq x_n \quad \forall\, n \geq H$ — (2)

by (1) & (2)

$$u - \epsilon/2 < x_H \leq x_n \leq u < u + \epsilon/2$$

$\Rightarrow \quad u - \epsilon/2 < x_n < u + \epsilon/2$ — (3) $\quad \forall\, n \geq H$

y

Let $m \geq H$, then we have

$$u - \epsilon/2 < x_m < u + \epsilon/2 \qquad \forall\, m \geq H$$

$$\Rightarrow -(u - \epsilon/2) > -x_m > -(u + \epsilon/2)$$

i.e. $\quad -u - \epsilon/2 < -x_m < -u + \epsilon/2$ — (4)

ow (3) + (4) $\Rightarrow$

$$-\epsilon < x_n - x_m < \epsilon$$

$\Rightarrow \quad |x_n - x_m| < \epsilon \quad \forall\, n, m \geq H$

$\Rightarrow (x_n)$ is a Cauchy seq.

Q.9 If $0<r<1$ & $|x_{n+1}-x_n| < r^n \quad \forall n \in \mathbb{N}$ — ①

Show that (x_n) is a Cauchy seq.

Sol^n: T.S. (x_n) is Cauchy, we need to show that for $\epsilon>0$, $\exists H \in \mathbb{N}$: $|x_n - x_m| < \epsilon \quad \forall n, m \geq H$

Now, $|x_n - x_m| = |\underbrace{x_n - x_{n+1}} + \underbrace{x_{n+1} - x_{n+2}} + x_{n+2} \cdots + \underbrace{x_{m-1} - x_m}|$

(let $m>n$)

$n \quad n+1 \quad \cdots \quad m$

$$\Rightarrow |x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$< r^n + r^{n+1} + \cdots + r^{m-1}$$

$$= r^n \left[1 + r + r^2 + \cdots + r^{-n+m-1}\right]$$

$$= r^n \left[1 + r + r^2 + \cdots + r^{m-(n+1)}\right]$$

$$= r^n \left[\frac{1(1 - r^{m-n})}{1-r}\right] < \frac{r^n}{1-r} \quad (m-n>0)$$

$\rightarrow 0$ as $n \rightarrow \infty$

$(\because 0 < r < 1)$

$\therefore |x_n - x_m| < \epsilon \quad \forall n, m \geq H$

$\Rightarrow (x_n)$ is a Cauchy seq.

Ex 3.5 (Bartle)

Q.3 (c) Show that $(x_n) = (\ln n)$ is not a Cauchy seq.

Sol^n: $x_n = \ln n$

$\Rightarrow x_m = \ln m$

Now, $|x_n - x_m| = |\ln n - \ln m| = |\ln \frac{n}{m}|$ (let $n>m$)

Let $n = 2m$.

$\therefore |x_n - x_m| = |\ln \frac{2m}{m}| = \ln 2$

Since $\ln 2$ can't be less than every $\epsilon > 0$

$\therefore$ for this choice of n & m, $|x_n - x_m| \not< \epsilon$ for every ϵ

$\therefore (x_n) = (\ln n)$ is not a Cauchy seq.

If (x_n) & (y_n) are two Cauchy sequences, show that (i) $(x_n + y_n)$ is a Cauchy seq.

(ii) $(x_n y_n)$ is a Cauchy seq.

(i) T.S. $z_n = x_n + y_n$ is a Cauchy seq.

i.e. T.S. $|z_n - z_m| < \epsilon \quad \forall\, n, m \geq H$

given: (x_n) & (y_n) are Cauchy sequences.

Since (x_n) is Cauchy $\therefore$ for $\epsilon > 0$, $\exists\, H_1 \in \mathbb{N}$:

$|x_n - x_m| < \epsilon/2 \quad \forall\, n, m \geq H_1$ —

so, (y_n) is Cauchy

$\therefore$ for $\epsilon > 0$, $\exists\, H_2 \in \mathbb{N}$:

$|y_n - y_m| < \epsilon/2 \quad \forall\, n, m \geq H_2$ —