SUB - NUMERICAL METHODS

CLASS - B.Sc. Physical science (III)rd year

"NUMERICAL DIFFERENTIATION"

It is the process of calculating the derivative of a f^n using numerical methods.

⇒ Derivatives using Newton's forward difference formula:-

Newton's forward difference formula:-

$$y = y_0 + h\Delta y_0 + \frac{h(h-1)}{2!}\Delta^2 y_0 + \frac{h(h-1)(h-2)}{3!}\Delta^3 y_0 + \cdots$$

Now Differentiate both sides w.r.t. h then

$$\frac{dy}{dh} = \Delta y_0 + \frac{2h-1}{2!}\Delta^2 y_0 + \frac{3h^2-6h+2}{3!}\Delta^3 y_0 + \cdots$$

We know that $x = x_0 + ih$, $h = \frac{x - x_0}{i}$ therefore $\frac{dh}{dx} = \frac{1}{i}$

Now $\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx}$.

$$\frac{dy}{dx} = \frac{1}{i}\left[\Delta y_0 + \frac{2h-1}{2!}\Delta^2 y_0 + \frac{3h^2-6h+2}{3!}\Delta^3 y_0 + \cdots\right] \longrightarrow (1)$$

At $x = x_0$, $h = 0$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{i}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right] \quad \text{--- } (2)$$

Again differentiating (1) w.r.t. x

we get.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dh}\right) \cdot \frac{dh}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{i}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6h-6}{3!}\Delta^3 y_0 + \frac{12h^2-36h+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{i}$$

Putting $h=0$, we get

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{i^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right]$$

Similarly

$$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{i^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

* <u>Derivative using Newton's Backward difference formula.</u>

<u>Newton's backward difference formula</u> :

$$y = y_n + h\nabla y_n + \frac{h(h+1)}{2!}\nabla^2 y_n + \frac{h(h+1)(h+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating both sides w.r.t. h, we get

$$\frac{dy}{dh} = \nabla y_n + \frac{2h+1}{2!}\nabla^2 y_n + \frac{3h^2+6h+2}{3!}\nabla^3 y_n + \cdots$$

$$\left[\because\ x = x_n + ih\right] \Rightarrow h = \frac{x-x_n}{i}$$

$$\frac{dh}{dx} = \frac{1}{i}$$

Now $$\frac{dy}{dx} = \frac{dy}{dh}\cdot\frac{dh}{dx}$$

$$= \frac{1}{i}\left[\nabla y_n + \frac{2h+1}{2!}\nabla^2 y_n + \frac{3h^2+6h+2}{3!}\nabla^3 y_n + \cdots\right]$$

At $x = x_n$, $h = 0$ hence putting $h = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{i}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad (1)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dh}\left(\frac{dy}{dx}\right)\frac{dh}{dx}$$

$$= \frac{1}{i^2}\left[\nabla^2 y_n + \frac{6h+6}{3!}\nabla^3 y_n + \frac{6h^2 + 18h + 11}{12}\nabla^4 y_n + \cdots\right]$$

Put $h = 0$, we get

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{i^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]$$

Similarly

$$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{i^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Q. 1.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	6.989	7.403	7.781	8.129	8.451	8.750	9.031

find the value $D'(x)$, $D''(x)$ at 1.1 and 1.6

Solⁿ :- Difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.0	6.989					
		0.414				
1.1	7.403		-0.036			
		0.378		0.006		
			-0.030		-0.002	.001
1.2	7.781			0.004		
		0.348			-0.001	
1.3	8.129		-0.026			
		0.322		0.003		.003
1.4	8.451		-0.023		0.002	
		0.299		0.005		
1.5	8.750		-0.018			
		0.281				
1.6	9.031					

We have

Newton forward difference formula

$$y = y_0 + h\Delta y_0 + \frac{h(h-1)}{2!}\Delta^2 y_0 + \frac{h(h-1)(h-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{i}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \rightarrow (1)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{i^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \rightarrow (2)$$

$\Rightarrow$ $i = 0.1$, $x_0 = 1.1$

Now putting the Value of forward operator in eqn ① and ②

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right]$$

$$= 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right]$$

$$= -3.74$$

for the value 1.6 we use Newton Backward difference formula

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{i}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \rightarrow (3)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{i^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \rightarrow (4)$$

here $i = 0.1$, $x_n = 1.6$

Putting the Value of Backward operator in eqn (3) and (4)

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right]$$

$$= 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(.002)\right]$$

$$= -0.714$$

Q. Find $f'(10)$ from the following table

$x =$	5	11	27	34
$f(x)$	23	899	17315	35606

Q. Given that

$x =$	1.2	1.3	1.4	1.5	1.6
$y =$	1.5095	1.6984	1.9043	2.1293	2.3756

find $f'(1.4)$ and $f''(1.4)$