Sub:- Ring theory and Linear Algebra

Chapter - 5

B.Sc (Maths) (H) III RD YR.
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Diagonalization (By FriedBerg)

★ **Diagonal Matrix** :- A square matrix A is said to be diagonal matrix if it is of the form $\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}_{n\times n}$

where $a_{ii} \in \mathbb{C}$ $\forall$ $i = 1, 2, \ldots, n$

★ **Definition** :- A linear operator T on a finite dimensional vector space V is called diagonalizable if there is an ordered basis β for V ~~st~~ such that $[T]_\beta$ is a diagonal matrix.

A square matrix A is called diagonalizable if L_A operator is diagonalizable

★ **Definition** :- Let $T : V \to V$ be a linear operator where V is a vector space. A vector $v (\neq 0) \in V$ is called an eigen vector of T if there exists a scalar λ such that $T(v) = \lambda v$.

The scalar λ is called the eigenvalue corresponding to the eigen vector v.

Let $A \in M_{n\times n}(F)$. A vector $v\ (\neq 0) \in F^n$ is called an eigen vector of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ.

The scalar λ is called the eigen value of A (or L_A) corresponding to the eigen vector v.

* The words 'characteristic' and 'Proper' are also used in place of 'Eigen'.

Theorem 5.1 A linear operator T on a finite dimensional vector space V is diagonizable $\Longleftrightarrow$ there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \le j \le n$.

Proof Let $T : V \to V$ be diagonizable and $\dim V = n$

$\Rightarrow$ there exists an ordered basis $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$ of V such that $[T]_\beta$ is a diagonal matrix.

Let $[T]_\beta = D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$

$\Rightarrow T\beta_i = a_{ii}\beta_i \quad \forall \; 1 \le i \le n$

Since β is an ordered basis for V $\Rightarrow$

$\beta_i \ne 0 \quad \forall \; 1 \le i \le n$

$\Rightarrow$ each β_i is an eigenvector of T.

$\Rightarrow$ β consists of eigenvectors of T.

Conversely, let there exists an ordered Basis β for V

consisting of eigenvectors of T. — ①

Since $\dim V = n$

$\Rightarrow$ order of $\beta = n$

let $\beta = \{v_1, v_2, \ldots, v_n\}$.

By eq^n ①, we have that each v_i is an eigen vector of T

$\Rightarrow$ for each $i = 1, 2, \ldots, n$, there exists $\lambda_i \in F$ such that $T v_i = \lambda_i v_i \quad \forall\ 1 \leq i \leq n$

$$\Rightarrow [T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} \equiv \text{diagonal Matrix.}$$

$\Rightarrow$ T is diagonalizable.

Ex 1 let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

Find eigen vectors, eigen values of A and $[L_A]_\beta$

Proof let $v = \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ be an eigen vector of A — (I)

$\Rightarrow$ There exists a $\lambda \in \mathbb{C}$ such that

$$Av = \lambda v$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 3y \\ 4x + 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x + 3y = \lambda x \\ 4x + 2y = \lambda y \end{array} \quad \Rightarrow \quad \begin{array}{l} (1-\lambda)x + 3y = 0 \\ 4x + (2-\lambda)y = 0 \end{array}$$

$$\Rightarrow (1-\lambda)x = -3y \quad - (2)$$

$$4x = (\lambda-2)y \quad - (3)$$

Multiplying by -3 ~~by~~, eq^n (3) becomes

$$-12x = (\lambda-2)(-3y) \quad - (4)$$

Substituting the value of $(-3y)$ from eq^n (2) ~~to~~ in eq^n (4)

$$\Rightarrow -12x = (\lambda-2)(1-\lambda)x$$

$$\Rightarrow -12x + (\lambda-2)(\lambda-1)x = 0$$

$$\Rightarrow x[(\lambda-2)(\lambda-1) - 12] = 0$$

~~eige~~ $\Rightarrow$ Either $x = 0$ or $(\lambda-2)(\lambda-1)-12 = 0$

If $x = 0$ then by eq^n (2), $y = 0$

$\Rightarrow v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ but $v \neq 0$ (see eq^n I)

$$\Rightarrow (\lambda-2)(\lambda-1) - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm \sqrt{49}}{2}$$

$$\lambda = \frac{3+7}{2}, \frac{3-7}{2} = 5, -2.$$

let $\lambda_1 = 5$ and $\lambda_2 = -2$

$\Rightarrow \{\lambda_1, \lambda_2\}$ are the eigenvectors of A

let $v_1 (\neq 0)$ be eigen vector of A corresponding to eigen value λ_1 and $v_1 = \begin{bmatrix} x_1 \\ \cdot \end{bmatrix}$

$$\Rightarrow Av_1 = \lambda_1 v_1$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 + 3y_1 &= 5x_1 \\ 4x_1 + 2y_1 &= 5y_1 \end{aligned}$$

$$\Rightarrow 3y_1 = 4x_1 \quad \& \quad 4x_1 = 3y_1$$

$$\Rightarrow v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 4/3\, x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 4/3 \end{bmatrix} ; x_1 \neq$$

Similarly, let $v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ is an eigenvector of A corresponding to eigen value $\lambda_2 = -2$

$$\Rightarrow Av_2 = \lambda_2 v_2 \Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_2 + 3y_2 &= -2x_2 \\ 4x_2 + 2y_2 &= -2y_2 \end{aligned}$$

$$\Rightarrow \begin{aligned} 3x_2 &= -3y_2 . \\ 2x_2 &= -2y_2 \end{aligned} \quad \Rightarrow x_2 = -y_2$$

$$\Rightarrow v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} ; x_2 \neq 0$$

★ Eigen space:- Let T be a Linear operator on a vector space V. Then the eigenspace of T corresponding to eigen value λ is

$$E_\lambda = \{ v : Tv = \lambda v \}$$

= {collection of all eigen vectors corresponding to

Therefore, for the above example,

$$E_5 = \left\{ x_1 \begin{bmatrix} 1 \\ 4/3 \end{bmatrix} ; x_1 \in \mathbb{C} \right\}$$

and $$E_{-2} = \left\{ x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} : x_2 \in \mathbb{C} \right\}$$

The collection E_5 and E_{-2} already contain 0 vector
$(\because$ if $x_1 = 0 \Rightarrow 0 \in E_5$ and if $x_2 = 0 \Rightarrow 0 \in E_{-2}$

Ex-3 let

$$C^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f^n\ f \text{ has derivatives of all orders} \}$$
~~equation~~ function

and $F(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a } f^n \}$

It is easy to show that $F(\mathbb{R}, \mathbb{R})$ is a vector space and $C^\infty(\mathbb{R})$ is its subspace.

Let $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be a f^n defined by

$$T(f) = f' \quad \forall\ f \in C^\infty(\mathbb{R}).$$

Find Eigenvalues and eigenvectors of T?

Proof To show T is a linear operator. (Prove it yourself)

Let $\lambda \in \mathbb{R}$ be an eigen value of T corresponding to eigen vector f $\Rightarrow f \neq 0$ and $T(f) = \lambda f$

$$\Rightarrow f' = \lambda f$$

$$\Rightarrow \frac{df}{f} = \lambda\, dt \quad \Rightarrow \int \frac{df}{f} = \lambda \int dt$$

$$\Rightarrow \log f = \lambda t + C_1$$ where C_1 is a constt of integration

$\Rightarrow \quad f(t) = e^{\lambda t + c} = Ce^{\lambda t}$ where $C = e^{c_1}$ is a constt.

$\in C^{\infty}(\mathbb{R})$.

$\Rightarrow$ For each real no. 'λ', there exists a non zero function f ($f \neq 0$ if $C \neq 0$) s.t

$$Tf = \lambda f$$

$\Rightarrow$ each real no. λ is an eigen ~~vector~~ value of T corresponding to eigen vector $Ce^{\lambda t}$ $(C \neq 0)$

Also, for $\lambda = 0$, $f(t) = C \neq 0$ if $C \neq 0$

$\Rightarrow \underbrace{\{\text{Eigen values of } T\}}_{\leftarrow \text{set}} = \mathbb{R}$.

Theorem 5.2 Let $A \in M_{n\times n}(F)$. Then a scalar λ is an eigen value of A iff $\Leftrightarrow \det(A - \lambda I_n) = 0$

Proof Let 'λ' be an eigenvalue of A

$\Leftrightarrow$ there exists a <u>non-zero</u> vector $v \in \mathbb{F}^n$ s.t

$$Av = \lambda v$$

$\Leftrightarrow \quad (A - \lambda I_n) v = 0$ for some $v (\neq 0)$ in $\mathbb{F}^n$ — (II)

$\Leftrightarrow$ Matrix $(A - \lambda I_n)$ is not invertible

$\because$ if $(A - \lambda I_n)$ is invertible $\Rightarrow (A - \lambda I_n)^{-1}$ exists such that $(A - \lambda I_n)^{-1}(A - \lambda I_n) = I_n$ — (III)

Consider $(A - \lambda I_n)^{-1}(A - \lambda I_n) v = (A - \lambda I_n)^{-1} 0$ ($\because$ eq^n II)

$= 0$

$\Rightarrow \quad I_n v = 0$ (By eq^n III)

$\Rightarrow \quad v = 0$ but $v \neq 0$.

$\Rightarrow \quad \det(A - \lambda I_n) = 0.$

<u>Definition</u> :- Let $A \in M_{n\times n}(\mathbb{F})$. The polynomial $f(t) = \det(A - \lambda I_n)$ is called <u>Characteristic polynomial</u> of A.

<u>Ex 4</u> Find eigen values of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

<u>Proof</u> Consider Characteristic polynomial of A,

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\therefore$ By theorem 5.2 says. 'λ' is an eigen value of A

$\Leftrightarrow \det(A - \lambda I_2) = 0$

$\Rightarrow \lambda = 3, -1$ are the eigen values of A.

<u>Definition</u> :- Let $T : V \to V$ be a linear operator where V = 'n' dim. vector space. with ordered basis β. The characteristic polynomial of $[T]_\beta$ is called the <u>characteristic polynomial</u> of T.

Notation :- Characteristic polynomial of T is denoted by ~~Tea~~ $\det(T - \lambda I)$.

Ex-5 Let $T : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ be a linear operator defined by $T(f(x)) = f(x) + (x+1)f'(x)$

Find Characteristic poly. of T ?

~~Proof~~ Sol^n We know that

$P^2(\mathbb{R}) \equiv$ set of polynomials over $\mathbb{R}$ of degree atmost 2.

$$= \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$$

$\beta = \{1, x, x^2\}$ forms a basis of $P^2(\mathbb{R})$.

$$T(1) = 1 + (x+1)\cdot 0 = 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = x + (x+1)\cdot 1 = 2x + 1$$

$$= 1 + 2\cdot x + 0\cdot x^2$$

$$T(x^2) = x^2 + (x+1)(2x) = 0 + 2x + 3\cdot x^2$$

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Characteristic polynomial of T is $\det\left[[T]_\beta - \lambda I_3 \right]$

$$= \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

~~e~~ ~~det~~

$$= \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

Hence by theorem 5.2, $\lambda = 1, 2, 3$ are the eigen values of T and

$f(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda)$ is the characteristic polynomial of T.

Do Exercise ~~Ques~~ ~~20~~ ~~&~~ Questions related to it.