Concepts of Gradient, Divergence, Curl and Related Problems

THE VECTOR DIFFERENTIAL OPERATOR DEL, written ∇, is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the *gradient*, the *divergence* and the *curl*. The operator ∇ is also known as *nabla*.

THE GRADIENT. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the *gradient* of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\nabla\phi\cdot\mathbf{a}$ and is called the *directional derivative* of ϕ in the direction $\mathbf{a}$. Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\mathbf{a}$.

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* V_1, V_2, V_3 .

SOLVED PROBLEMS

THE GRADIENT

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\,\mathbf{k}$$

$$= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k}$$

$$= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$$

3. Find $\nabla\phi$ if (*a*) $\phi = \ln|\mathbf{r}|$, (*b*) $\phi = \frac{1}{r}$.

(*a*) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|\mathbf{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$.

$$\nabla\phi = \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2)$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

(*b*)
$$\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\{(x^2+y^2+z^2)^{-1/2}\}$$

$$= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \mathbf{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \mathbf{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \mathbf{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\}$$

$$= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

4. Show that $\nabla r^n = nr^{n-2}\mathbf{r}$.

$$\nabla r^n = \nabla(\sqrt{x^2+y^2+z^2})^n = \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{j}\frac{\partial}{\partial y}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{k}\frac{\partial}{\partial z}\{(x^2+y^2+z^2)^{n/2}\}$$

$$= \mathbf{i}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2x\} + \mathbf{j}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2y\} + \mathbf{k}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2z\}$$

$$= n(x^2+y^2+z^2)^{n/2-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= n(r^2)^{n/2-1}\mathbf{r} = nr^{n-2}\mathbf{r}$$

Note that if $\mathbf{r} = r\mathbf{r}_1$ where $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, then $\nabla r^n = nr^{n-1}\mathbf{r}_1$.

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Let $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$ or $(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$

i.e. $\nabla\phi\cdot d\mathbf{r} = 0$ so that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

$\nabla(x^2y + 2xz) = (2xy + 2z)\mathbf{i} + x^2\mathbf{j} + 2x\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ at the point $(2,-2,3)$.

Then a unit normal to the surface $= \dfrac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2+(4)^2+(4)^2}} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

Another unit normal is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ having direction opposite to that above.

7. Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$.

$$\nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4)\mathbf{i} - 3x\mathbf{j} + 4xz\mathbf{k}$$

Then a normal to the surface at the point $(1,-1,2)$ is $7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$.

The equation of a plane passing through a point whose position vector is $\mathbf{r}_0$ and which is perpendicular to the normal $\mathbf{N}$ is $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{N} = 0$. (See Chap.2, Prob.18.) Then the required equation is

$$[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})]\cdot(7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$$

or

$$7(x-1) - 3(y+1) + 8(z-2) = 0.$$

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\nabla\phi = \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$
$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi \cdot \mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \cdot (\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, ϕ is increasing in this direction.

11. (*a*) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum? (*b*) What is the magnitude of this maximum?

$$\nabla\phi = \nabla(x^2yz^3) = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$$
$$= -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} \quad \text{at} \quad (2,1,-1).$$

Then by Problem 9,

(*a*) the directional derivative is a maximum in the direction $\nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$,

(*b*) the magnitude of this maximum is $|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176} = 4\sqrt{11}$.

12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,-1,2)$.

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2 + y^2 + z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2 + y^2 - 3$ or $x^2 + y^2 - z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2 + y^2 - z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta$, where θ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}| \cos\theta$$
$$16 + 4 - 4 = \sqrt{(4)^2 + (-2)^2 + (4)^2}\,\sqrt{(4)^2 + (-2)^2 + (-1)^2}\,\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \text{arc}\cos 0.5819 = 54^\circ 25'$.

THE DIVERGENCE

15. If $\mathbf{A} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla\cdot\mathbf{A}$ (or div $\mathbf{A}$) at the point $(1,-1,1)$.

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \quad \text{at } (1,-1,1).$$

16. Given $\phi = 2x^3y^2z^4$. (*a*) Find $\nabla\cdot\nabla\phi$ (or div grad ϕ).
(*b*) Show that $\nabla\cdot\nabla\phi = \nabla^2\phi$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operat

(*a*) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k}$$

Then $$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

(*b*) $$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

17. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{and} \quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

The equation $\nabla^2\phi = 0$ is called *Laplace's equation.* It follows that $\phi = 1/r$ is a solution of this equation.

THE CURL

23. If $\mathbf{A} = xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k}$, find $\nabla\times\mathbf{A}$ (or curl $\mathbf{A}$) at the point (1,−1,1).

$$\nabla\times\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\mathbf{k}$$

$$= (2z^4 + 2x^2y)\,\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

24. If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl $\mathbf{A}$.

$$\text{curl curl }\mathbf{A} = \nabla\times(\nabla\times\mathbf{A})$$

$$= \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \nabla\times\left[(2x+2z)\mathbf{i} - (x^2+2z)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2z & 0 & -x^2-2z \end{vmatrix} = (2x+2)\mathbf{j}$$

27. Prove: (*a*) $\nabla\times(\nabla\phi) = \mathbf{0}$ (curl grad $\phi = \mathbf{0}$), (*b*) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ (div curl $\mathbf{A} = 0$).

(*a*)
$$\nabla\times(\nabla\phi) = \nabla\times\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\mathbf{k}$$

$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k} = \mathbf{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

(*b*)
$$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} = 0$$

assuming that $\mathbf{A}$ has continuous second partial derivatives.

Note the similarity between the above results and the results $(\mathbf{C}\times\mathbf{C}m) = (\mathbf{C}\times\mathbf{C})m = \mathbf{0}$, where m is a scalar and $\mathbf{C}\cdot(\mathbf{C}\times\mathbf{A}) = (\mathbf{C}\times\mathbf{C})\cdot\mathbf{A} = 0$.

29. Prove $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$.

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \times \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k} \right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) \right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \right]\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_3}{\partial z\,\partial y} + \frac{\partial^2 A_1}{\partial x\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z}\right)\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_1}{\partial x\,\partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \mathbf{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 \mathbf{A} + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$