Vector Integration

ORDINARY INTEGRALS OF VECTORS. Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$. If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u. The *definite integral* between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\,\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

This integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral*. If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In aerodynamics and fluid mechanics this integral is called the *circulation* of $\mathbf{A}$ about C, where $\mathbf{A}$ represents the velocity of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

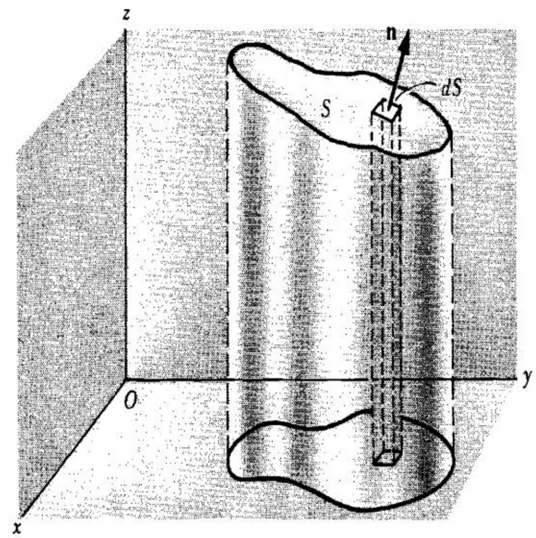
For methods of evaluation of line integrals, see the Solved Problems.

The following theorem is important.

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of the positive side of S is called a *positive* or *outward drawn* unit normal.

Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\,dS$. The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$



is an example of a surface integral called the *flux* of $\mathbf{A}$ over S. Other surface integrals are

$$\iint_S \phi\,dS, \quad \iint_S \phi\,\mathbf{n}\,dS, \quad \iint_S \mathbf{A}\times d\mathbf{S}$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus (see Problem 17).

The notation $\oint\!\!\oint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may also be used.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point. However, this does not pose any real problem since we can generally subdivide S into surfaces which do satisfy this restriction.

VOLUME INTEGRALS. Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \mathbf{A}\,dV \quad \text{and} \quad \iiint_V \phi\,dV$$

are examples of *volume integrals* or *space integrals* as they are sometimes called. For evaluation of such integrals, see the Solved Problems.

LINE INTEGRALS

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

   (*a*) $x = t,\ y = t^2,\ z = t^3$.

   (*b*) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

   (*c*) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(*a*) If $x=t,\ y=t^2,\ z=t^3$, points (0,0,0) and (1,1,1) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2-28t^6+60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(*b*) Along the straight line from (0,0,0) to (1,0,0) $y=0,\ z=0,\ dy=0,\ dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \left(3x^2+6(0)\right)dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) $x=1,\ z=0,\ dx=0,\ dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2+6y)0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x = 1,\ y = 1,\ dx = 0,\ dy = 0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \left(3(1)^2+6(1)\right) 0 \ - \ 14(1)\, z(0) \ + \ 20(1)\, z^2\, dz \ = \ \int_{z=0}^{1} 20\, z^2\, dz \ = \ \frac{20\, z^3}{3}\Big|_0^1 \ = \ \frac{20}{3}$$

Adding,
$$\int_C \mathbf{A}\cdot d\mathbf{r} \ = \ 1 + 0 + \frac{20}{3} \ = \ \frac{23}{3}$$

(c) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x = t,\ y = t,\ z = t$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} \ = \ \int_{t=0}^{1} (3t^2+6t)\, dt \ - \ 14(t)(t)\, dt \ + \ 20(t)(t)^2\, dt$$

$$= \ \int_{t=0}^{1} (3t^2+6t-14t^2+20t^3)\, dt \ = \ \int_{t=0}^{1} (6t-11t^2+20t^3)\, dt \ = \ \frac{13}{3}$$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$.

$$\text{Total work} \ = \ \int_C \mathbf{F}\cdot d\mathbf{r} \ = \ \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \ \int_C 3xy\, dx \ - \ 5z\, dy \ + \ 10x\, dz$$

$$= \ \int_{t=1}^{2} 3(t^2+1)(2t^2)\, d(t^2+1) \ - \ 5(t^3)\, d(2t^2) \ + \ 10(t^2+1)\, d(t^3)$$

$$= \ \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\, dt \ = \ 303$$

8. If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

Since the integration is performed in the xy plane (z=0), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \ = \ \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \ \int_C 3xy\, dx \ - \ y^2\, dy$$

First Method. Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t,\ y = 2t^2$. Points (0,0) and (1,2) correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \ = \ \int_{t=0}^{1} 3(t)(2t^2)\, dt \ - \ (2t^2)^2\, d(2t^2) \ = \ \int_{t=0}^{1} (6t^3 - 16t^5)\, dt \ = \ -\frac{7}{6}$$

Second Method. Substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \ = \ \int_{x=0}^{1} 3x(2x^2)\, dx \ - \ (2x^2)^2\, d(2x^2) \ = \ \int_{x=0}^{1} (6x^3 - 16x^5)\, dx \ = \ -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 instead of $-7/6$.

10. (*a*) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(*b*) Conversely, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

(*a*) Work done $= \displaystyle\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x,y,z)$ is single-valued at all points P_1 and P_2.

(*b*) Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. By hypothesis, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points, which we take as (x_1, y_1, z_1) and (x,y,z) respectively. Then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1dx + F_2dy + F_3dz$$

is independent of the path joining (x_1, y_1, z_1) and (x,y,z). Thus

$$\phi(x+\Delta x, y, z) - \phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} - \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(x,y,z)}^{(x_1,y_1,z_1)} \mathbf{F}\cdot d\mathbf{r} + \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(x,y,z)}^{(x+\Delta x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$$

Since the last integral must be independent of the path joining (x,y,z) and $(x+\Delta x, y, z)$, we may choose the path to be a straight line joining these points so that dy and dz are zero. Then

$$\frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\,dx$$

Taking the limit of both sides as $\Delta x \to 0$, we have $\dfrac{\partial\phi}{\partial x} = F_1$.

Similarly, we can show that $\dfrac{\partial\phi}{\partial y} = F_2$ and $\dfrac{\partial\phi}{\partial z} = F_3$.

Then $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi$.

If $\displaystyle\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining P_1 and P_2, then $\mathbf{F}$ is called a *conservative field*. It follows that if $\mathbf{F} = \nabla\phi$ then $\mathbf{F}$ is conservative, and conversely.

Proof using vectors. If the line integral is independent of the path, then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\,ds$$

By differentiation, $\dfrac{d\phi}{ds} = \mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$. But $\dfrac{d\phi}{ds} = \nabla\phi\cdot\dfrac{d\mathbf{r}}{ds}$ so that $(\nabla\phi - \mathbf{F})\cdot\dfrac{d\mathbf{r}}{ds} = 0$.

Since this must hold irrespective of $\dfrac{d\mathbf{r}}{ds}$, we have $\mathbf{F} = \nabla\phi$.

SURFACE INTEGRALS.

17. Give a definition of $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$ over a surface S in terms of limit of a sum.

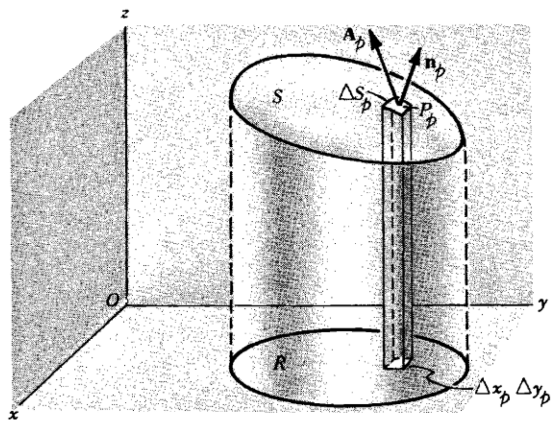
Subdivide the area S into M elements of area ΔS_p where $p = 1,2,3,\ldots, M$. Choose any point P_p within ΔS_p whose coordinates are (x_p, y_p, z_p). Define $\mathbf{A}(x_p, y_p, z_p) = \mathbf{A}_p$. Let $\mathbf{n}_p$ be the positive unit normal to ΔS_p at P. Form the sum

$$\sum_{p=1}^{M} \mathbf{A}_p\cdot\mathbf{n}_p\, \Delta S_p$$

where $\mathbf{A}_p\cdot\mathbf{n}_p$ is the normal component of $\mathbf{A}_p$ at P_p.

Now take the limit of this sum as $M\to\infty$ in such a way that the largest dimension of each ΔS_p approaches zero. This limit, if it exists, is called the surface integral of the normal component of $\mathbf{A}$ over S and is denoted by

$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$



18. Suppose that the surface S has projection R on the xy plane (see figure of Prob.17). Show that

$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS \quad = \quad \iint_R \mathbf{A}\cdot\mathbf{n}\, \frac{dx\, dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

By Problem 17, the surface integral is the limit of the sum

$$(1) \qquad \sum_{p=1}^{M} \mathbf{A}_p\cdot\mathbf{n}_p\, \Delta S_p$$

The projection of ΔS_p on the xy plane is $|(\mathbf{n}_p\, \Delta S_p)\cdot\mathbf{k}|$ or $|\mathbf{n}_p\cdot\mathbf{k}|\, \Delta S_p$ which is equal to $\Delta x_p \Delta y_p$ so that $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\mathbf{n}_p\cdot\mathbf{k}|}$. Thus the sum (1) becomes

$$(2) \qquad \sum_{p=1}^{M} \mathbf{A}_p\cdot\mathbf{n}_p \frac{\Delta x_p \Delta y_p}{|\mathbf{n}_p\cdot\mathbf{k}|}$$

By the fundamental theorem of integral calculus the limit of this sum as $M\to\infty$ in such a manner that the largest Δx_p and Δy_p approach zero is

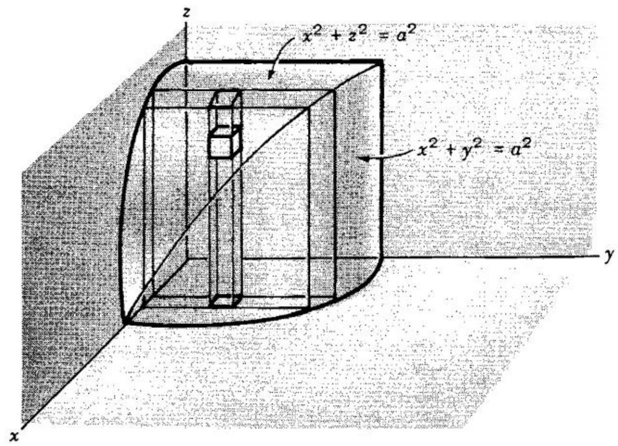
$$\iint_R \mathbf{A}\cdot\mathbf{n}\, \frac{dx\, dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

and so the required result follows.

Strictly speaking, the result $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\mathbf{n}_p\cdot\mathbf{k}|}$ is only approximately true but it can be shown on closer examination that they differ from each other by infinitesimals of order higher than $\Delta x_p \Delta y_p$, and using this the *limits* of (1) and (2) can in fact be shown equal.

Volume Integrals

27. Find the volume of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.



Required volume = 8 times volume of region shown in above figure

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx \quad = \quad 8\int_{x=0}^{a} (a^2-x^2)\,dx \quad = \quad \frac{16a^3}{3}$$