Weekly e-material (30/3/2020 to 3/4/2020)
(24)
Name :- Konthoujam Somorjit Singh
paper code :- II 4 E 1
paper Title :- Numerical Methods
class :- physics (H) 2nd year.

Romberg's Method

Romberg's method use Richardson's extrapolation in a system symmetric way. In trapezoidal rule, the error is of order 2 and spacing ratio $\frac{h_1}{h_2} = 2$. then from Richardson extrapolation, we have,

$$R = \frac{4}{3} Q_2 - \frac{1}{3} Q_1$$

$$= Q_2 + \frac{Q_2 - Q_1}{3}$$

For Simpson's rule, we have error in R of order 4.

Taking $\frac{h_1}{h_2} = 2$, we get

$$S = \frac{16}{15} R_2 - \frac{1}{15} R_1$$

$$= \left(\frac{1}{15} + 1\right) R_2 - \frac{1}{15} R_1$$

$$= R_2 + \frac{R_2 - R_1}{15}$$

For Cote's formula $n=4$, i.e, error is of order 6.

Taking $\frac{h_1}{h_2} = 2$, we get

$$T = \frac{64}{63} S_2 - \frac{1}{63} S_1$$

$$= S_2 + \frac{S_2 - S_1}{63}.$$

Q) Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x}$ correct to four decimal places.

→ Let $h = \frac{1}{2}$, therefore the value of integral $f(x) = \frac{1}{1+x}$ are

x	$y = f(x)$
0	1
$1/2$	0.6666
1	0.5

Therefore, by trapezoidal rule, we have

$$Q_1 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right]$$

$$= \frac{1}{4}\left[1 + 2(0.6666) + 0.5\right]$$

$$= \frac{1}{4}(2.8332)$$

$$= 0.7083$$

$1.526 + 1.5864 + 1.6094) + 1.6486]$

$= 1.8276.$

5) Use Romberg's method to compute $\int_4^{5.2} \log x \, dx$ from the data.

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
$y = \log_e x$:	1.3863	1.4351	1.4816	1.526	1.5686	1.6094	1.6486

→ Here $h = 0.4$. ($\because h = x_{i+1} - x_i$).

Then, by trapezoidal rule

$$Q_1 = \frac{h}{2}\left[y_0 + 2(y_1 + y_2) + y_3 \ldots\right]$$

$$= \frac{0.4}{2}\left[1.3863 + 2(1.4816 + 1.5686) + 1.6486\right]$$

$$= 1.8271.$$

Now, let $h = 0.2$.

Then, again by trapezoidal rule.

$$Q_2 = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5\right]$$

$$= \frac{0.2}{2}\left[1.3863 + 2(1.4351 + 1.4816 + 1.526 + 1.526 + 1.5686 + 1.6094) + 1.6486\right]$$

$$= 1.8276.$$

Now, let $h = \frac{1}{4}$. Then the values of the integrand are:

x	y
0	1
$\frac{1}{4}$	0.8
$\frac{1}{2}$	0.6666
$\frac{3}{4}$	0.5714
1	0.5

Therefore, by Trapezoidal rule,

$$I_2 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3) + y_4\right]$$

$$= \frac{1}{8}\left[1 + 2(0.8 + 0.6666 + 0.5714) + 0.5\right]$$

$$= 0.697.$$

Then, by Romberg method, we get

$$R_2 = \frac{4}{3} I_2 - \frac{1}{3} I_1$$

$$= \frac{4}{3}(0.6971) - \frac{1}{3}(0.7083)$$

$$= 0.6932.$$

$R_1 = \frac{4}{3} Q_2 - \frac{1}{3} Q_1$

$= \frac{4}{3}(1.8276) - \frac{1}{3}(1.8271)$

$= 1.82776.$

Q) Solve using Romberg method

$$\int_0^1 \frac{1+x}{1+x^3} dx.$$

→

x	0	0.5	1
$y = f(x)$	1	4/3	1

Take $h = 1$, we get

By Trapezoidal rule, we have

$Q_1 = \frac{h}{2}[y_0 + y_1]$

$= \frac{1}{2}[1+1] = 1.$

Take $h = \frac{1}{2}$, by Trapezoidal rule, we have

$Q_2 = \frac{h}{2}[y_0 + 2y_1 + y_2]$

$= \frac{1}{4}[1 + 2 \cdot \frac{4}{3} + 1]$

$= \frac{1}{4} \times \frac{14}{3} = \frac{14}{12}$

Then, by Romberg's method, we have

$$R_1 = \frac{4B_2 - Q_1}{3}$$ (29)

$$= \frac{4 \times \frac{14}{12} - 1}{3}$$

$$= \frac{11}{9} = 1.222$$

Quadrature.

Gaussian Quadrature

The general formula for Gaussian quadrature is

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{2} c_i f(x_i)$$

Gaussian quadrature will only provide good result if the function $f(x)$ is well approximated by a polynomial function with the range $[-1, 1]$. Here value of c_i and x_i depends on the choice of x.

If we choose the quadrature points $x_1, x_2, \ldots x_n$ as the zeros of nth degree Lagrange polynomial and by using correct coefficient, the formula is exact for polynomial of degree upto $2n-1$. The Gaussian quadrature rule for two evaluation points which is exact upto degree 3 polynomial,

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$
$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

similarly, Gaussian formula for 3-evaluation point which is exact upto 5-degree polynomial, has the form.

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$
$$= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

Q) Solve the integral $\int_{-1}^{1} e^{x^2}\,dx$ using Gaussian quadrature.

$\rightarrow$ $$\int_{-1}^{1} e^{x^2}\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$
$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= e^{\left(-\frac{1}{\sqrt{3}}\right)^2} + e^{\left(\frac{1}{\sqrt{3}}\right)^2}$$
$$= 2e^{\frac{1}{3}} = 2.13954$$
$$= 4.27908$$