Weekly e-material (23/3/2020 to 27/3/2020) (10)

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Error term in quadrature formula

Since, we know that function f is approximated by a polynomial of degree n in Quadrature formula i.e.

$$f(x) \approx P_n(x).$$

On integration, we have

$$\int_{x_0}^{x_n} f(x)dx \approx \int_{x_0}^{x_n} P_n(x)dx \quad -(1)$$

If $R_n(x)$ is taken as difference between f(x) and $P_n(x)$ then

$$f(x) = P_n(x) + R_n(x).$$

Therefore, $\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} (P_n(x) + R_n(x))dx.$

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} P_n(x)dx + \int_{x_0}^{x_n} R_n(x)dx$$

Hence, error between true value of integral and the value of eqn. (1) is

$$\int_{x_0}^{x_n} P_n(x)\,dx.$$

For n degree polynomial, we have

$$R_n(x) = \frac{f^{n+1}(\xi)\,\pi(x)}{(n+1)!}; \text{ where}$$

$$\pi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

and $\xi \in (x_0, x_n)$.

Now consider equi-spaced ordinates, then

$x = x_0 + uh$ & $x_n = x_0 + nh$.

$$x - x_0 = uh$$

$$x - x_1 = (x_0 + uh) - (x_0 + h)$$
$$= (u-1)h.$$

$$x - x_2 = x_0 + uh - (x_0 + 2h)$$
$$= (u-2)h.$$

$$\vdots$$

$$x - x_n = (u-n)h.$$

We get

$$\pi(x) = uh(u-1)h(u-2)h\cdots(u-n)h$$
$$= h^{n+1}[u(u-1)(u-2)\cdots(u-n)]$$

putting value of $\pi(x)$ in $R_n(x)$, we get. (12)

$$R_n(x) = h^{n+1} \frac{f(\xi)}{(n+1)!} u^{(n+1)},$$

where $u^{(n+1)} = u(u-1)(u-2)\ldots(u-n)$

Therefore, error equation in quadrature formula is

$$E_n(x) = \int_{x_0}^{x_n} R_n(x)\,dx$$

$$= \frac{h}{(n+1)!} \int_0^n h^{n+1} f^{n+1}(\xi)\, u^{(n+1)}\, du$$

$$= \frac{h^{n+2}}{(n+1)!} \int_0^n f^{n+1}(\xi)\, u^{(n+1)}\, du.$$

$$= \frac{h^{n+2} f^{n+1}(\xi)}{(n+1)!} \int_0^n u^{(n+1)}\, du$$

$E_n(x)$ can be written in two forms

$$E_n(x) = \frac{h^{n+2} f^{n+1}(\xi)}{(n+1)!} \int_0^n u^{(n+1)}\, du$$ when n is odd

$$E_n(x) = \frac{h^{n+3} f^{n+2}(\xi)}{(n+1)!} \int_0^n \left(u - \frac{n}{2}\right) u^{(n+1)}\, du$$

when n is even.

(1) For $n=1$ (odd), Trapezoidal rule is obtained. i.e.

$$E_1(x) = \frac{h^3}{2!} f''(\xi) \int_0^1 u(u-1)\,du.$$

$$= \frac{h^3}{2!} f''(\xi) \left[\frac{u^3}{3} - \frac{u^2}{2}\right]_0^1$$

$$= -\frac{h^3 f''(\xi)}{12}, \quad x_0 < \xi < x_1.$$

Therefore, the error in the trapezoidal rule becomes

$$E_1 = -\frac{h^3 f''(\xi)}{12}, \quad x_0 < \xi < x_1.$$

(2) If $n=2$ (even), Simpson's rule is obtained.

The error is given by

$$E_2(x) = \frac{h^5 f^{(iv)}(\xi)}{4!} \int_0^2 u(u-1)(u-2)\,du.$$

$$= -\frac{h^5}{90} f^{(iv)}(\xi), \quad x_0 < \xi < x_2.$$

Summing up for $n/2$ intervals,

$$E_n = -\frac{n}{2}\frac{h^5}{90} f^{(iv)}(\xi)$$

$$= \frac{-(nh)h^4}{180} f^{(iv)}(\xi)$$

$$= \frac{-(x_n - x_0) h^4}{180} f^{(iv)}(\xi) \;, \quad x_0 < \xi < x_n. \qquad (14)$$

Simpson's rule has error of order 4.

Therefore, Complete Simpson's formula is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[f_0 + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2}) + f_n\Big] - \frac{(x_n - x_0)}{180} h^4 f^{(iv)}(\xi).$$

Remark :- Trapezoidal rule for solving Integration is

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + f(b)\right]$$
$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right].$$

Example :- Evaluate $\int_0^1 x \sin(\pi x)\,dx$ for $h = 1$.

$$\rightarrow \int_0^1 x \sin(\pi x)\,dx = \frac{1}{2}\left[f(0) + f(1)\right]$$
$$= \frac{1}{2} \sin \pi = 0$$

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Richardson extrapolation (Deferred approach of limit)

Let Q be true value of a derivative or integral Q_1 and Q_2 be approximate value of Q.

Let h_1 and h_2 be the spacing. If order of error is n, truncating the error series after its first term we obtain:

$$Q - Q_1 \approx C.h_1^n$$

$$Q - Q_2 \approx C h_2^n$$, where C is Constant.

$$\frac{Q - Q_1}{h_1^n} = \frac{Q - Q_2}{h_2^n} \approx C.$$

$$\Rightarrow Q\left(\frac{1}{h_1^n} - \frac{1}{h_2^n}\right) = \frac{Q_1}{h_1^n} - \frac{Q_2}{h_2^n}.$$

$$Q_{12} \approx \frac{\left(\frac{h_1}{h_2}\right)^n Q_2 - Q_1}{\left(\frac{h_1}{h_2}\right)^n - 1}$$, which is h^n extrapolation formula of Richardson.

This formula gives approximate value Q_{12} of Q.

Generally, we consider the case of

$$\frac{h_1}{h_2} = 2.$$

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In trapezoidal rule, order of error is 2.

Hence, extrapolation formula with $\frac{h_1}{h_2} = 2$ becomes,

$$Q_{12} \approx \frac{2^2 Q_2 - Q_1}{2^2 - 1} = \frac{4}{3} Q_2 - \frac{1}{3} Q_1,$$

which is $\frac{1}{3} h^2$ extrapolation formula.

In Simpson's rule the order of error is 4.

The extrapolation formula becomes $\frac{1}{15} h^4$ extrapolation given by

$Q_{12} \approx \frac{16}{5} Q_2 - \frac{1}{15} Q_1$ with error $O(h^4)$ and $\frac{h_1}{h_2} = 2$.

Q) Evaluate $\int_1^2 \frac{dx}{x}$ by

(a) Simpson's rule with $h = 0.50$

(b) Simpson's rule with $h = 0.25$

(c) Richardson's extrapolation.

Compare the results obtained with the exact value

→ Take, $h = \frac{1}{2} = 0.5$

The value of x & y are tabulated below:-

x	y
1	1
$1+h=\frac{3}{2}$	$\frac{2}{3}$
$1+2h=2$	$\frac{1}{2}$.

Trapezoidal rule gives:-

$$\int_1^2 \frac{1}{x}\,dx = \frac{h}{3}\left[y_0+4y_1+y_2\right]$$

$$= \frac{1}{6}\left[1+4\cdot\frac{2}{3}+\frac{1}{2}\right]$$

$$= \frac{1}{6}\left[1+\frac{8}{3}+\frac{1}{2}\right]$$

$$= 0.69444.$$

(b) Take $h=\frac{1}{4}=0.25$

The value of x & y are tabulated below:-

x	y
1	1
$1+h=\frac{5}{4}$	$\frac{4}{5}$
$1+2h=\frac{3}{2}$	$\frac{2}{3}$
$1+3h=\frac{7}{4}$	$4/7$
$1+4h=2$	$\frac{1}{2}$

(18)

$$\int_1^2 \frac{1}{x}dx = \frac{h}{3}\cdot\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$= \frac{1}{12}\left[1 + 4\left(\frac{4}{5} + \frac{4}{7}\right) + 2\cdot\frac{2}{3} + \frac{1}{2}\right]$$

$$= 0.69324.$$

(c) We have $\frac{h_1}{h_2} = \frac{0.50}{0.25} = 2.$

Also, Simpson's rule is of order 4.

Therefore, $\theta_{12} \approx \frac{16}{15}(0.69324) - \frac{1}{15}(0.69444) = 0.69316.$

That is in good agreement with exact value.

$\log 2 - \log 1 = 0.69315.$

Newton Cote's formula

Newton Cote's formula are group of formulas for numerical integration, for evaluting the integrand at equally spaced points.

Newton Cote's formulae can be useful if the values of the integrand are at equally spaced points.

It is assume that the value of function 'f' defined on interval [a, b] is known to be at equally spaced points x_i for $i = 0, 1, 2, \ldots, n$, where $x_0 = a$ & $x_n = b$.

There are two types of Newton' Cote's formulae:

(1) The 'closed' type Newton Cote's formula which uses the function value at all points.

(2) The 'open' type Newton Cote's formula which does not uses the function value at the end points

(i) Newton Cote's open formulas.

(a) The Mid-point rule :

In mid-point rule, we use only single function evaluation, at the midpoint of the interval,

$$x_m = \frac{(a+b)}{2}.$$

$$\int_a^b f(x)\,dx = (b-a)\, f(x_m) + \frac{(b-a)^3}{24} f''(\xi_1),$$

$\xi_1 \in (a,b)$.

The error is given by

$$\frac{(b-a)^3}{24} f''(\xi).$$

(b) Two-point formula :-

Two-point formula uses two function evaluations.

$$x_0 = a \; ; \; x_2 = \frac{a+2b}{3}$$

$$x_1 = \frac{2a+b}{3} \; ; \; x_3 = b.$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(x_1) + f(x_2)\right]$$

or including the error term, we have

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(x_1) + f(x_2)\right] + \frac{(b-a)^3}{36} f''(\eta)$$

for some $\eta \in (a,b)$.

(c) The three-point formula :-

The three-point Newton Cote's open formula use three function evaluation, we define

$x_0 = a$, $x_2 = \frac{2a+2b}{4}$, $x_3 = \frac{a+3b}{4}$

$x_1 = \frac{3a+b}{4}$, $x_4 = b$

Taking $h = \frac{b-a}{4}$, we have

$$\int_a^b f(x)dx \approx \frac{4h}{3}\left[2f(x_1) - f(x_2) + 2f(x_3)\right]$$

Using error term,

$$\int_a^b f(x)dx = \frac{4h}{3}\left[2f(x_1) - f(x_2)\right] + \left(\frac{14h^5}{45}\right) f^{(4)}(\eta),$$

$\eta \in (a,b)$.

Example 1 :- Use the two - point formula to approximate the integral

$$\int_0^{\pi/2} \frac{\sin x}{2x}\, dx.$$

$\rightarrow$ Here $x_0 = a = 0$.

$$x_1 = \frac{2a+b}{3} = \frac{\pi}{6}$$

$$x_2 = \frac{a+2b}{3} = \frac{\pi}{3}$$

$$x_3 = \frac{\pi}{2}.$$

We know that in two- point method

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(x_1) + f(x_2)\right]$$

$$= \frac{\pi}{6}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{2\left(\frac{\pi}{2}\right)} + \frac{\sin\left(\frac{\pi}{3}\right)}{2\left(\frac{\pi}{3}\right)}\right]$$

$$= (0.045534) \times 45$$

$$= 2.04903.$$

(2) Use the three-point rule to approximate the interval for the following integral:

$$\int_0^2 \frac{dx}{x^2}$$

→ Here $x_0 = a = 0$

$$x_1 = \frac{3a+b}{4} = \frac{3\cdot 0+2}{4} = \frac{1}{2}$$

$$x_2 = \frac{2a+2b}{4} = \frac{2\cdot 0+2\cdot 2}{4} = 1$$

$$x_3 = \frac{a+3b}{4} = \frac{3\cdot 2}{4} = 1.5$$

$$x_4 = b = 2.$$

$$\& \; h = \frac{b-a}{4} = \frac{2-0}{4} = \frac{1}{2}$$

We have,

$$\int_a^b f(x)\,dx \approx \frac{4h}{3}\left[2f(x_1) - f(x_2) + 2f(x_3)\right]$$

$$\approx \frac{4(1)}{2(3)}\left[2\left(\frac{1.4}{1}\right) - 1 + 2\frac{1}{(1.5)^2}\right]$$

$$= \frac{4}{6}\left[2.8 - 1 + \frac{2}{2.25}\right]$$

$$= \underline{\underline{5.25926}}.$$