5 Normal Distribution

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5.1 Normal Distribution

(a). One of the most important examples of a continuous probability distribution is the Normal distribution, sometimes called the Gaussian distribution.

(b). Like Poisson distribution, Normal distribution is also a special case of Binomial distribution.

(c). Number of trials, $n \to \infty$ and probability of success, $p \to 1/2$.

(d). Important integral, $I = \int_0^\infty x^n\, e^{-\alpha\, x^m}\, dx = \frac{1}{m\,\alpha^{(n+1)/m}}\Gamma\left(\frac{n+1}{m}\right)$

where, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n+1) = \begin{cases} n\,\Gamma(n), & n \text{ is a fraction} \\ n!, & n \in Z^+ \end{cases}$

(e). The density function for Normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\,\sigma^2}, \qquad -\infty \le x \le \infty \tag{1}$$

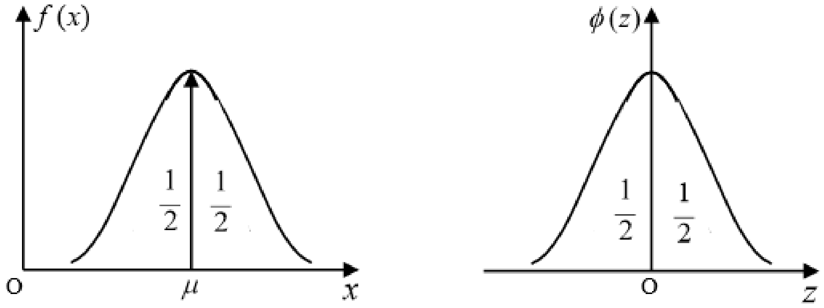
here, μ and σ are the mean and standard deviation respectively.

The curve $y = f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\,\sigma^2}$ is bell-shaped and symmetrical about the line $x = \mu$ and is called the normal curve, shown in Fig. 5.1(a).

The corresponding distribution function is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(\nu)\; d\nu \tag{2}$$

If X has the distribution function listed above in eq. (2), then we say that the random variable X is normally distributed with mean μ and variance σ^2



(a) The Normal Probability Curve (b) Standard Normal Probability Curve

Fig. 5.1: Probability Curves

5.2 Standard Normal distribution

If we put $Z = \frac{X-\mu}{\sigma}$, then Z is called **standard variable** corresponding to X.
The mean or expected value of Z can be obtained from eq. (1):

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \tag{3}$$

this is often referred to as the standard normal density function.
The curve $y = \phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$ is also bell-shaped and symmetrical about the line $z = 0$ and is called the normal curve, shown in Fig. 5.1(b).

The corresponding distribution function is given by

$$\Phi(z) = P(0 \leq Z \leq z) = \int_0^z \phi(u)\, du \tag{4}$$

Note: $\int_0^{z_1} \phi(u)\, du$ represents the area bounded by the curve,

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \quad \text{and z-axis} \quad (0 \leq z \leq z_1),$$

shown in Fig. 5.2.

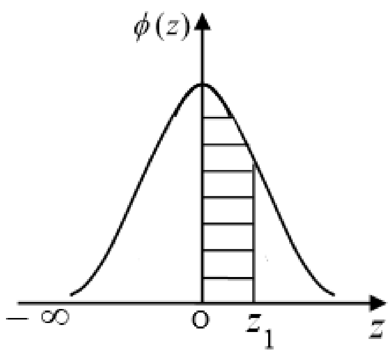


Fig. 5.2: Area between 0 and z_1.

5.3 Mean of Normal Ditribution

Mean of Normal distribution is defined as

$$\begin{aligned}
\bar{x} &= \int_{-\infty}^{\infty} x\, f(x) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-\mu)^2/2\sigma^2}\, dx && [\because\ f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}] \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\, z + \mu)\, e^{-z^2/2}\, \sigma\, dz && [\text{Where } z = \frac{x-\mu}{\sigma}] \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-z^2/2}\, dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \\
&= \frac{\sigma}{\sqrt{2\pi}}\,(0) + \frac{\mu}{\sqrt{2\pi}}\,\sqrt{2\pi} && [\because\ \int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}] \\
&= \mu
\end{aligned}$$

Note: $\bar{x}$ is also known as first moment, denoted by μ_1.

5.4 Second Moment of Normal Distribution

Second Moment of Normal distribution is given by

$$\begin{aligned}
\bar{x^2} &= \int_{-\infty}^{\infty} x^2\, f(x) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2\, e^{-(x-\mu)^2/2\sigma^2}\, dx && [\because\ f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}] \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\, z + \mu)^2\, e^{-z^2/2}\, \sigma\, dz && [\text{Where } z = \frac{x-\mu}{\sigma}] \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma^2\, z^2 + 2\,\sigma\,\mu\, z + \mu^2)\, e^{-z^2/2}\, \sigma\, dz \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{-z^2/2}\, dz + \frac{2\,\mu\,\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-z^2/2}\, dz + \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\,\sqrt{2\pi} + \frac{2\,\mu\,\sigma}{\sqrt{2\pi}}\,(0) + \frac{\mu^2}{\sqrt{2\pi}}\sqrt{2\pi} && [\because\ \int_{-\infty}^{\infty} u^2\, e^{-u^2/2}\, du = \sqrt{2\pi},\ \int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}] \\
&= \sigma^2 + \mu^2
\end{aligned}$$

5.5 Variance of Normal Distribution

Variance of Normal distribution is given by

$$\begin{aligned}
Variance &= \bar{x^2} - \bar{x}^2 \\
&= \sigma^2 + \mu^2 - \mu^2 \\
&= \sigma^2
\end{aligned}$$

5.6 Standard Deviation of Normal Distribution

Standard Deviation of Normal Distribution is given by

$$\text{standard deviation} = \sqrt{\text{variance}} = \sqrt{\sigma^2} = \sigma$$

5.7 Some Important Results

- $\Phi(\infty) = 1/2$.
- If Z is standard normal random variable

$$P(a \leq Z \leq b) = \Phi(b) - \Phi(a)$$

- If X is normally distributed with $\mu, \sigma \neq 0$, then $Z = \frac{X-\mu}{\sigma}$ has **standard** normal distribution.

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$$

- $P(Z \geq a) = \Phi(\infty) - P(0 \leq Z \leq a)$

5.8 Examples

Example 1. Prove that:

(a) $P(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} \phi(u)\, du$ Where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

Solution We have $\Phi(z) = P(Z \leq z) = \int_0^z \phi(u)\, du$

$$\begin{aligned}
\therefore \qquad P(z_1 \leq Z \leq z_2) &= \Phi(z_2) - \Phi(z_1) \\
&= \int_0^{z_2} \phi(u)\, du - \int_0^{z_1} \phi(u)\, du \\
&= \int_0^{z_2} \phi(u)\, du + \int_{z_1}^{0} \phi(u)\, du \\
&= \int_{z_1}^{z_2} \phi(u)\, du
\end{aligned}$$

The area is shown in Fig. 5.3.

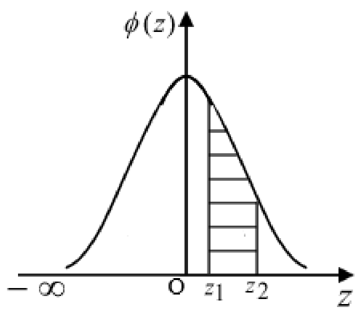


Fig. 5.3: Area between z_1 and z_2.

(b) $P(-z \le Z \le 0) = P(0 \le Z \le z)$

Solution We have $\Phi(z) = P(Z \le z) = \int_0^z \phi(u)\, du$

$$\therefore \quad P(-z \le Z \le 0) = \int_{-z}^{0} \phi(u)\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-z}^{0} e^{-u^2/2}\, du \qquad [\because \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}]$$

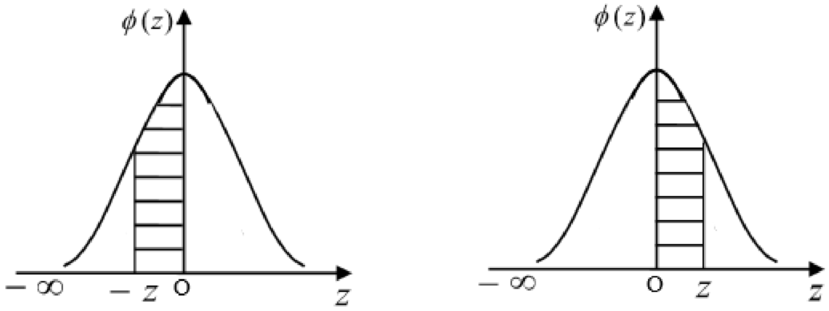
$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{0} e^{-t^2/2}\, (-dt) \qquad [\text{putting } u = -t]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-t^2/2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-u^2/2}\, du$$

$$= P(0 \le Z \le z)$$

i.e., $\Phi(-z) = -\Phi(z)$, This is shown in Fig. 5.4(a) and Fig. 5.4(b).



(a) Area between $-z$ and 0. (b) Area between 0 and z.

Fig. 5.4: Area under the curve

(c) $P(Z \le -z) = P(Z \ge z)$

Solution We have $\Phi(z) = P(Z \le z) = \int_0^z \phi(u)\, du$

$$\therefore \quad P(Z \le -z) = \int_{-\infty}^{-z} \phi(u)\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-u^2/2}\, du \qquad [\because \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}]$$

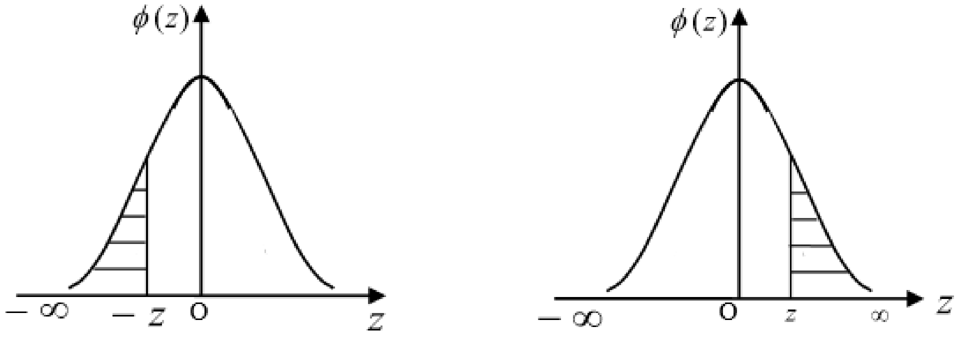
$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{z} e^{-t^2/2}\, (-dt) \qquad [\text{putting } u = -t]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-t^2/2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2}\, du$$

$$= P(Z \ge z)$$

This is shown in Fig. 5.5(a) and Fig. 5.5(b).



(a) Area between $-\infty$ and $-z$. (b) Area between z and ∞.

Fig. 5.5: Area under the curve

Example 2. Suppose the diameters d of bolts manufactured by a company are normally distributed with mean $\mu = 0.25$ inches and standard deviation $\sigma = 0.02$ inches. A bolt is considered defective if $d \leq 0.20$ inches or $d \geq 0.28$ inches. Find the percentage of defective bolts manufactured by the company.

Solution Given $\mu = 0.25$ inches and $\sigma = 0.02$ inches. We find $P(0.20 \leq d \leq 0.28)$,

$$\begin{aligned}
\therefore \qquad P(0.20 \leq d \leq 0.28) &= P\left(\frac{0.20-\mu}{\sigma} \leq \frac{d-\mu}{\sigma} \leq \frac{0.28-\mu}{\sigma}\right) \\
&= P\left(\frac{0.20-0.25}{0.02} \leq \frac{d-\mu}{\sigma} \leq \frac{0.28-0.25}{0.02}\right) \\
&= P\left(\frac{-0.05}{0.02} \leq \frac{d-\mu}{\sigma} \leq \frac{0.03}{0.02}\right) \\
&= P\left(-2.5 \leq \frac{d-\mu}{\sigma} \leq 1.5\right) \\
&= \Phi(1.5) - \Phi(-2.5) \\
&= \Phi(1.5) + \Phi(2.5) \qquad [\because\ \Phi(-b) = -\Phi(b)] \\
&= 0.4332 + 0.4938 \\
&= 0.927
\end{aligned}$$

i.e., The probability that the bolt is **not** defective, is 0.927. Therefore the probability that the bolt is defective is 0.073. Hence the the percentage of defective bolts manufactured by the company is **7.3 %**.

Example 3. Find the probability that 200 tosses of a coin will result in (a) between 80 and 120 heads inclusive, (b) less than 90 heads.

Solution This is a binomial experiment $B(n,p)$ with $n = 200$, $p = 1/2$ and $q = 1-p = 1/2$. Then

$$\mu = np = 200(1/2) = 100, \qquad \sigma = \sqrt{npq} = \sqrt{200(1/2)(1/2)} = \sqrt{50} \approx 7.07$$

Let X denotes the number of times the head occurs.

(a) We have to find $P(80 \leq X \leq 120)$ or, assuming the data are continuous, $P(79.5 \leq X \leq 120.5)$, then

$$\begin{aligned} P(79.5 \leq X \leq 120.5) &= P\left(\frac{79.5 - 100}{7.07} \leq \frac{X - 100}{7.07} \leq \frac{120.5 - 100}{7.07}\right) \\ &= P\left(-2.90 \leq \frac{X - 100}{7.07} \leq 2.90\right) \\ &= 2 \times \Phi(2.90) \\ &= 2 \times 0.4981 \\ &= 0.9962 \end{aligned}$$

(b) We have to find $P(0 \leq X < 90)$ or, assuming the data are continuous, $P(X \leq 89.5)$, then

$$\begin{aligned} P(X \leq 89.5) &= P\left(\frac{X - 100}{7.07} \leq \frac{89.5 - 100}{7.07}\right) \\ &= P\left(\frac{X - 100}{7.07} \leq -1.48\right) \\ &= P\left(\frac{X - 100}{7.07} \geq 1.48\right) \\ &= 0.5 - P\left(0 \leq \frac{X - 100}{7.07} \leq 1.48\right) \\ &= 0.5 - \Phi(1.48) \\ &= 0.5 - 0.4306 \\ &= 0.0694 \end{aligned}$$

Example 4. Find the probability of getting more than 25 "sevens" in 100 tosses of a pair of fair dice.

Solution This is a binomial experiment $B(n, p)$ with $n = 100$, $p = 1/6$ and $q = 1 - p = 5/6$. Then

$$\mu = n\,p = 100(1/6) = 16.67, \qquad \sigma = \sqrt{n\,p\,q} = \sqrt{100(1/6)(5/6)} = \sqrt{500/36} \approx 3.73$$

Let X denotes the number of times the "seven" appears.

We have to find $P(X > 25)$ or, assuming the data are continuous, $P(X \geq 25.5)$, then

$$\begin{aligned} P(X \geq 25.5) &= P\left(\frac{X - 16.67}{3.73} \geq \frac{25.5 - 16.67}{3.73}\right) \\ &= P\left(\frac{X - 16.67}{3.73} \geq 2.38\right) \\ &= 0.5 - P\left(0 \leq \frac{X - 16.67}{3.73} \leq 2.38\right) \\ &= 0.5 - \Phi(2.38) \\ &= 0.5 - 0.4913 \\ &= 0.0087 \end{aligned}$$

5.9 Problems

1. Let Z be the standard normal random variable. Find
   (a) $P(-0.81 \leq Z \leq 1.13)$
   (b) $P(Z \leq 0.73)$
   (c) $P(|Z| \leq 0.25)$

2. Find z_1 if $P(Z \geq z_1) = 0.84$, where z is normally distributed with mean 0 and variance 1.

3. Let X be normally distributed with mean $\mu = 8$ and standard deviation $\sigma = 4$. Find
   (a) $P(10 \leq X \leq 15)$
   (b) $P(X \leq 5)$

4. Suppose the weights of 2000 male students are normally distributed with mean $\mu = 155$ pounds and standard deviation $\sigma = 20$ pounds. Find the number of students with weights:
   (a) less than or equal to 100 pounds
   (b) between 120 & 130 pounds inclusive
   (c) greater than or equal to 200 pounds

5. If a set of measurements are normally distributed, what percentage of these differ from the mean by (a) more than half the standard deviation, (b) less than three quarters of the standard deviation?

6. A fair coin is tossed 10 times. Find the probability of obtaining between 4 and 7 heads inclusive of using:
   (a) the binomial distribution,
   (b) the normal approximation to the binomial distribution

7. If μ is the mean and σ is the standard deviation of a set of measurements that are normally distributed, what percentage of the measurements are (a) within the range $\mu \pm 2\,\sigma$ (b) outside the range $\mu \pm 1.2\,\sigma$(c) greater than $\mu - 1.5\,\sigma$?

8. Find the probability that a student can guess correctly the answers to
   (a) 12 or more out of 20
   (b) 24 or more out of 40, questions in a true-false examination.

9. Assume that 4 percent of the population over 65 years old has Alzheimer's disease. Suppose a random sample of 3500 people over 65 is taken. Find the probability P that fewer than 150 of them have the disease.

10. The mean grade on a final examination was 72, and the standard deviation was 9. The top 10 percent of the students are to receive A's. What is the minimum grade a student must get in order to receive an A?

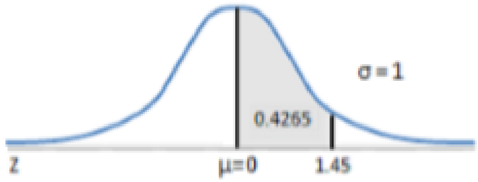
5.10 References

(a). Introduction to Probability and Statistics, 2016 by Seymour Lipschutz and John J. Schiller.

(b). Probability and Statistics, 3e by Murray R. Spiegel, John J. Schiller and R. Alu Srinivasan.

Areas Under the One-Tailed Standard Normal Curve

This table provides the area between the mean and some Z score.
For example, when Z score = 1.45 the area = 0.4265.



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000