Dr. Ram Pravesh Prasad (1) Linear Algebra
GE-2, Sem-2
Section-3

Orthogonality

Orthogonal and Orthonormal Vectors:-

① **Definition** :- Let $\{v_1, v_2, \ldots v_k\}$ be a subset of k-distinct vectors of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots v_k\}$ is an orthogonal set of vectors iff the dot product of any two distinct vectors in this set is zero. i.e. $v_i \cdot v_j = 0$ for $1 \le i, j \le k$, $i \ne j$. Also $\{v_1, v_2, \ldots v_k\}$ is an orthonormal set of vectors iff it is orthogonal set and all its vectors are unit vectors [i.e $\|v_i\| = 1$], for $1 \le i \le k$.

In particular, any set containing a single vector is orthogonal, and any set containing a single unit vector is orthonormal.

~~Ex:- In $\mathbb{R}^3$ $\{i, j, k\}$ is an orthogonal set~~ because $i \cdot j = j \cdot k = k \cdot i = 0$. In fact this is an orthonormal set, since we also have

$$\|i\| = \|j\| = \|k\| = 1.$$

~~In $\mathbb{R}^4$, $\{(1, 0, -1), 1\}$~~

In $\mathbb{R}^4$, $\{(1,0,-1,0), (3,0,3,0)\}$ is an orthogonal set because $(1,0,-1,0)\cdot(3,0,3,0) = 0$. if we normalize each vector ~~&~~ (i.e. divide each of these vectors by its length); we create the orthonormal set of vectors

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)\right\}$$

Theorem:- ~~~~ Let $T = \{v_1, v_2, \cdots v_k\}$ be an orthogonal set of nonzero vectors in $\mathbb{R}^n$. Then T is a linearly independent set.

Orthogonal and orthonormal bases:

Definition:- A basis B for a subspace W of $\mathbb{R}^n$ is an orthogonal ~~basis~~ basis for W iff B is an orthogonal set. Similarly, a basis B for W is an orthonormal basis for W iff B is an orthonormal set.

Result:- If B is an orthogonal set of n nonzero vectors in $\mathbb{R}^n$, then B is an orthogonal basis of $\mathbb{R}^n$. Similarly if B is an orthonormal set of n vectors in $\mathbb{R}^n$, then B is an orthonormal basis ~~of~~ for $\mathbb{R}^n$.

Ex:- Consider $S = \{(1,0,-1), (-1,4,-1), (2,1,2)\}$ be a subset of $\mathbb{R}^3$, verify that given set is orthogonal and also find orthonormal basis.

Solve:- Given that

$$S = \{(1,0,-1), (-1,4,-1), (2,1,2)\}$$

$$(1,0,-1)\cdot(-1,4,-1) = -1+0+1 = 0$$

$$(-1,4,-1)\cdot(2,1,2) = -2+4-2 = 0$$

$$(2,1,2)\cdot(1,0,-1) = 2+0-2 = 0$$

hence the given set is orthogonal set hence by above result S is orthogonal basis

For orthonormal set, normalise each vector of S. hence

$$W = \left\{ \frac{(1,0,-1)}{\sqrt{2}}, \frac{(-1,4,-1)}{3\sqrt{2}}, \frac{(2,1,2)}{3} \right\}$$

W is orthonormal set hence it is orthonormal basis.

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Theorem: If $B = \{(v_1, v_2, \ldots v_k)\}$ is a nonempty ordered orthogonal basis for a subspace W of $\mathbb{R}^n$, and if v is any vector in W, then

$$[v]_B = \left[\frac{(v \cdot v_1)}{(v_1 \cdot v_1)}, \frac{(v \cdot v_2)}{(v_2 \cdot v_2)}, \ldots \frac{(v \cdot v_k)}{(v_k \cdot v_k)}\right]$$

$$= \left[\frac{(v \cdot v_1)}{\|v_1\|^2}, \frac{(v \cdot v_2)}{\|v_2\|^2}, \ldots, \frac{(v \cdot v_k)}{\|v_k\|^2}\right]$$

In particular, if B is an ordered orthonormal basis for W, then $[v]_B = [v \cdot v_1, v \cdot v_2, \ldots, v \cdot v_k]$.

Ex:- Consider $B = \{(1,0,-1), (-1,4,-1), (2,1,2)\}$ is an orthogonal basis of $\mathbb{R}^3$. Let $v = (-1,5,3)$. find $[v]_B$.

Solve:- We know that by previous result

$$[v]_B = \left[\frac{v \cdot v_1}{\|v_1\|^2}, \frac{v \cdot v_2}{\|v_2\|^2}, \ldots \frac{v \cdot v_k}{\|v_k\|^2}\right]$$

$$= \left(\frac{-4}{2}, \frac{10}{10}, \frac{9}{9}\right) = (-2, 1, 1)$$

Ex:- Consider $W = \left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)\right\}$ be an orthonormal basis of $\mathbb{R}^3$. Let $v = (-1,5,3)$ then find ~~$[v]_{BV}$~~ $[v]_W$.

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~~The Gram Schmidt~~

The Gram-Schmidt Process for Finding an Orthogonal Basis for a Subspace of $\mathbb{R}^n$.

Suppose W is a subspace of $\mathbb{R}^n$ with $B = \{w_1, w_2, \ldots w_k\}$. There is a straightforward way to replace B with an orthogonal basis for W. This is known as the Gram-Schmidt process

Gram-Schmidt Process

Let $\{w_1, w_2, \ldots w_k\}$ be a linearly independent subset of $\mathbb{R}^n$. We create a new set $\{v_1, v_2, \ldots v_k\}$ of vectors as follows.

Let $v_1 = w_1$

Let $v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$.

Let $v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{w_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$\vdots$

Let $v_k = w_k - \left(\frac{w_k \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{w_k \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{w_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}}\right) v_{k-1}$

Then $\{v_1, v_2, \ldots v_k\}$ is an orthogonal basis for span$\{(w_1, w_2, \ldots w_k)\}$.

Theorem:- Let $B = \{w_1, w_2, \ldots w_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{v_1, v_2, \ldots v_k\}$ obtained by applying the Gram-Schmidt process to B is an ~~orthogona~~ orthogonal basis for W.

Hence, any non trivial subspace W of $\mathbb{R}^n$ has an orthogonal basis.

Ex:- Let $B = \{(2,1,0,-1), (1,0,2,-1), (0,-2,1,0)\}$ be a subset $\mathbb{R}^4$. Let $w_1 = (2,1,0,-1)$, $w_2 = (1,0,2,-1)$ and $w_3 = (0,-2,1,0)$. replace B as an orthogonal basis by applying Gram-schmidt process.

& Let $W = $ span(B).

Solve:- by Gram-Schmidt process

$$v_1 = w_1 = (2,1,0,-1)$$

$$v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= (1,0,2,-1) - \left(\frac{(1,0,2,-1)\cdot(2,1,0,-1)}{(2,1,0,-1)\cdot(2,1,0,-1)}\right)(2,1,0,-1)$$

$$v_2 = (1,0,2,-1) - \left(\frac{3}{6}\right)(2,1,0,-1) = \left(0, -\frac{1}{2}, 2, -\frac{1}{2}\right)$$

$$v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{w_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= (0,-2,1,0) - \left(\frac{(0,-2,1,0)\cdot(2,1,0,-1)}{(2,1,0,-1)\cdot(2,1,0,-1)}\right)(2,1,0,-1)$$

$$- \left(\frac{(0,-2,1,0)\cdot(0,-1,4,-1)}{(0,-1,4,-1)\cdot(0,-1,4,-1)}\right)(0,-1,4,-1)$$

$$= (0,-2,1,0) - \left(\frac{-2}{6}\right)(2,1,0,-1) - \frac{6}{18}(0,-1,4,-1)$$

$$= \left(\frac{2}{3}, \frac{-4}{3}, \frac{-1}{3}, 0\right) = (2,-4,-1,0)$$

hence $\{v_1, v_2, v_3\} = \{(2,1,0,-1), (0,-1,4,-1), (2,-4,-1,0)\}$
is an orthogonal basis for W

To find orthonormal basis, normalize v_1, v_2 & v_3
we obtain

$$\left\{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{-1}{\sqrt{6}}\right), \left(0, \frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right)\right.$$
$$\left.\left(\frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, 0\right)\right\}$$

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Orthogonal Complements

Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement, $W^\perp$, of W in $\mathbb{R}^n$ is the set of all vectors $x \in \mathbb{R}^n$ with the property that $x \cdot w = 0$, for all $w \in W$. That is, $W^\perp$ contains those vectors of $\mathbb{R}^n$ orthogonal to every vector in W.

Theorem:- If W is a subspace of $\mathbb{R}^n$, then $v \in W^\perp$ iff v is orthogonal to every vector in a spanning set for W.

Ex:- Consider the subspace $W = \{(a,b,0) \mid a,b \in \mathbb{R}\}$ of $\mathbb{R}^3$. find $W^\perp$

Solve:- Given that

$$W = \{(a,b,0) \mid a,b \in \mathbb{R}\}$$

Since W is spanned by $\{(1,0,0),(0,1,0)\}$

$W^\perp$ = Set of all vector of $\mathbb{R}^3$ which are orthogonal to $(1,0,0), (0,1,0)$

if $(x,y,z) \in \mathbb{R}$

then $(x,y,z)\cdot(1,0,0) = 0$

$(x,y,z)\cdot(0,1,0) = 0$

i.e $x = 0$

$y = 0$

hence $W^{\perp} = \{(0,0,z) \mid z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

$\dim(W^{\perp}) = 1$

$\dim(W) = 2$

$\dim(W^{\perp}) + \dim(W) = \dim(\mathbb{R}^3)$

$1 + 2 = 3$

Ex!- Consider the subspace

$W = \{a(-3,2,4) \mid a \in \mathbb{R}\}$ of $\mathbb{R}^3$

find $W^{\perp} = ?$

Solve! — Try yourself

Properties:- (*) Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$, and

$$W \cap W^\perp = \{0\}$$

(*) Let W be a subspace of $\mathbb{R}^n$. Let $\{v_1, \ldots v_k\}$ be an orthogonal basis for W contained in an orthogonal basis $\{v_1, v_2, \ldots v_k, v_{k+1}, \ldots v_n\}$ for $\mathbb{R}^n$. Then $\{v_{k+1}, \ldots v_n\}$ is an orthogonal basis for $W^\perp$

(*) Let W be subspace of $\mathbb{R}^n$. Then

$$\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n) = n$$

(*) Let W be a subspace of $\mathbb{R}^n$. Then $(W^\perp)^\perp = W$

~~orthogonal projection onto a subsp~~

Orthogonal projection onto a subspace

Result:- Let W be a subspace of $\mathbb{R}^n$. Then every vector $v \in \mathbb{R}^n$ can be expressed in a unique way as $w_1 + w_2$, where $w_1 \in W$, and $w_2 \in W^\perp$.

Definition:- Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, u_2, \ldots u_k\}$ and let $v \in \mathbb{R}^n$. Then the orthogonal projection of v on W is the vector

$$\text{Proj}_W v = (v \cdot u_1) u_1 + \cdots + (v \cdot u_k) u_k$$

of W is the trivial subspace of $\mathbb{R}^n$

Then $\text{Proj}_W v = 0$

Ex:- Consider the orthonormal subset

$$B = \{u_1, u_2\} = \left\{ \left(\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right), \left(\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right) \right\}$$

of $\mathbb{R}^3$. Let $v = (1, 2, 3)$ then find $\text{Proj}_W v = ?$ where $W = \text{span}(B)$

Solve:- ~~Given that~~

Given that $B = \left\{ \left(\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right), \left(\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right) \right\}$

is an orthonormal set, and hence it is an orthonormal basis (Verify)

and $v = (1, 2, 3)$ & $W = span(B)$

by previous result

$$Proj_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$$

$$= -\frac{2}{3}\left(\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right) + \frac{11}{3}\left(\frac{4}{9}, \frac{4}{7}, \frac{7}{7}\right)$$

$$= \left(\frac{28}{27}, \frac{46}{27}, \frac{85}{27}\right)$$

~~* If W is a subspace of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then~~ v can be expressed as $w_1 + w_2$, where ~~$w_1 = Proj_W v$~~ $w_1 = Proj_W v \in W$ and $w_2 = v - Proj_W v \in W^\perp$.

moreover, w_2 can also be expressed as $Proj_{W^\perp} v$.

(*) Let W be subspace of $\mathbb{R}^n$. Then the mapping $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L(v) = \text{Proj}_W(v)$ is a linear operator with $\ker(L) = W^{\perp}$

(*) Let W be a subspace of $\mathbb{R}^n$, and assume all vectors in W have initial points at the origin. Let P be any point in n-dimensional space. Then the minimum distance from P to W is the shortest distance between P and the terminal point of any vector in W.

(*) i.e. If P be a point in n-dimensional space and v is the vector from origin to P, then the minimum distance from P to W is $\|v - \text{Proj}_W v\|$.

Ex! - Let W be a subspace of $\mathbb{R}^3$, whose vectors lie in the plane $2x + y + z = 0$. For $v = (-6, 10, 5)$ we ~~can~~ calculate $v - \text{Proj}_W v = (1, \frac{1}{2}, \frac{1}{2})$ & the minimum distance from $P = (-6, 10, 5)$ to W is

$$\|v - \text{Proj}_W v\| = \sqrt{\left(1 + \frac{1}{4} + \frac{1}{4}\right)} = \sqrt{\frac{3}{2}} \simeq 1.2247$$

Ans

Que: Let W be the subspace of $\mathbb{R}^3$ whose vectors lie in the plane $2x+y+z=0$. Let $v=(-6,10,5)$ find $Proj_W v$

Solve: $W=\{2x+y+z=0 \mid (x,y,z)\in\mathbb{R}^3\}$

hence W consist two lenearly independent vector, which obtained by putting

$\cancel{x=1,y}$ $(y=1,\ z=0 \mid y=0,\ z=0)$

For $y=1,\ z=0$, then $x=-\frac{1}{2}$

~~for $y=0\ z=1$, then~~ $x=-\frac{1}{2}$

$(-1,2,0),(-1,0,2)$

hence vectors $\{(-1,2,0),(-1,0,2)\}$

~~Apply Gram Schmidt process~~

$u_1=v_1=(-1,2,0)$

$$u_2=v_2-\frac{(v_2\cdot u_1)}{u_1\cdot u_1}u_1$$

$$=(-1,0,2)-\frac{(-1,0,2)\cdot(-1,2,0)}{\sqrt{5}}(-1,2,0)$$

$=$

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$$u_2 = (-1,0,2) - \frac{(1+0+0)}{\sqrt{5}}(-1,2,0)$$

$$= (-1,0,2) - \frac{(-1,2,0)}{\sqrt{5}}$$

hence $\{u_1, u_2\}$ are orthogonal basis
Normalizing these vectors we get ~~orthogon~~ orthonormal basis

$$\{v_1, v_2\}$$

hence $Proj_W v = (v \cdot v_1)v_1 + (v \cdot v_2)v_2$

finally $v - Proj_W v = ?$

hence we decompose the vector $v = (-6,10,5)$ into sum of two vectors $Proj_W v$ & $v - Proj_W v$.