OPTICAL ELECTRONICS

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The general analysis presented above (Eqs. (11.101)–(11.111)) is rigorously valid for TE modes in planar waveguides. For TM modes and for modes in cylindrical waveguides, the orthonormality condition becomes more involved. Indeed, for guided modes the rigorously correct orthogonality condition is given by (see, e.g., Snyder and Love (1983), Chapter 11)

$$\iint [\mathbf{E}_m(x, y) \times \mathbf{H}_k^*(x, y) \cdot \hat{z}]\, \mathrm{d}x\, \mathrm{d}y = \iint [\mathbf{E}_k(x, y) \times \mathbf{H}_m^*(x, y) \cdot \hat{z}]\, \mathrm{d}x\, \mathrm{d}y = 0 \quad \text{for } k \neq m \tag{11.126}$$

where the integral is over the *entire* transverse cross section; the refractive index distribution $n^2(x, y)$ can be arbitrary and the modal field is defined by the equation

$$\mathscr{E}_m(x, y, z, t) = \mathbf{E}_m(x, y)\, \mathrm{e}^{\mathrm{i}(\omega t - \beta_m z)} \tag{11.127}$$

and a similar equation for **H**; m represents the mode number. However, in the weakly guiding approximation, which is valid for almost *all* practical waveguides, we can still use the orthonormality condition given by Eq. (11.104) where u_m is assumed to represent the transverse component of the electric (or magnetic) field. In this approximation therefore, the entire analysis for the calculation of the excitation coefficient etc. [Eqs. (11.101)–(11.111)] remains valid.

Problem 11.5: Starting from Eq. (11.66) show that the TM modes satisfy the following orthogonality condition:

$$\int_{-\infty}^{+\infty} (1/n^2) H_y^{(m)}(x) H_y^{(k)*}(x)\, \mathrm{d}x = 0 \qquad m \neq k$$

Show that the above equation is consistent with Eq. (11.126).
(Hint: Use Eq. (11.12))

Problem 11.6: Consider a symmetric planar waveguide with $n_1 = 1.5$, $n_2 = 1.496$ operating at $\lambda_0 = 1\ \mu$m with the fundamental mode having a propagation constant of $\beta/k_0 = 1.498$. (*a*) Calculate the penetration depth $1/\gamma$ in the cover. (*b*) Obtain the angle at which the rays representing the mode are travelling in the waveguide. (*c*) Calculate the width of the waveguide.
(Answer: (*a*) 2.06 μm, (*b*) 2.96°, (*c*) $\approx 3.2\ \mu$m).

11.9 Maxwell's equations in inhomogeneous media: TE and TM modes in planar waveguides

In this section we will derive the equations which are the starting points for modal analysis. We start with Maxwell's equations, which for an isotropic, linear, non-conducting and nonmagnetic medium take the form

$$\nabla \times \mathscr{E} = -\partial \mathscr{B}/\partial t = -\mu_0 \partial \mathscr{H}/\partial t \tag{11.128}$$

$$\nabla \times \mathscr{H} = \partial \mathscr{D}/\partial t = \epsilon_0 n^2 \partial \mathscr{E}/\partial t \tag{11.129}$$

$$\nabla \cdot \mathscr{D} = 0 \tag{11.130}$$

$$\nabla \cdot \mathscr{B} = 0 \tag{11.131}$$

where we have used the constitutive relations

$$\mathscr{B} = \mu_0 \mathscr{H} \tag{11.132}$$

$$\mathscr{D} = \epsilon \mathscr{E} = \epsilon_0 n^2 \mathscr{E} \tag{11.133}$$

in which $\mathscr{E}$, $\mathscr{D}$, $\mathscr{B}$ and $\mathscr{H}$ represent the electric field, electric displacement, magnetic induction and magnetic intensity respectively, $\mu_0 (= 4\pi \times 10^{-7}\ \mathrm{N\,s^2/C^2})$ represents the free space magnetic permeability, $\epsilon [= \epsilon_0 K = \epsilon_0 n^2]$ represents the dielectric permittivity of the medium, K and n are respectively the dielectric constant and the refractive index and ϵ_0 $[= 8.854 \times 10^{-12}\ \mathrm{C^2/N\,m^2}]$ is the permittivity of free space. Now taking the curl of Eq. (11.128) and using Eq. (11.129) we get

$$\nabla \times (\nabla \times \mathscr{E}) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \mathscr{H}) = -\mu_0 \epsilon_0 n^2 \frac{\partial^2 \mathscr{E}}{\partial t^2}$$

or

$$\nabla(\nabla \cdot \mathscr{E}) - \nabla^2 \mathscr{E} = -\epsilon_0 \mu_0 n^2 \frac{\partial^2 \mathscr{E}}{\partial t^2} \tag{11.134}$$

Further

$$0 = \nabla \cdot \mathscr{D} = \epsilon_0 \nabla \cdot (n^2 \mathscr{E}) = \epsilon_0 [\nabla n^2 \cdot \mathscr{E} + n^2 \nabla \cdot \mathscr{E}]$$

Thus

$$\nabla \cdot \mathscr{E} = -(1/n^2) \nabla n^2 \cdot \mathscr{E} \tag{11.135}$$

Substituting in Eq. (11.134) we obtain

$$\nabla^2 \mathscr{E} + \nabla\left(\frac{1}{n^2} \nabla n^2 \cdot \mathscr{E}\right) - \epsilon_0 \mu_0 n^2 \frac{\partial^2 \mathscr{E}}{\partial t^2} = 0 \tag{11.136}$$

The above equation shows that for an inhomogeneous medium the equations for $\mathscr{E}_x$, $\mathscr{E}_y$ and $\mathscr{E}_z$ are coupled. For a homogeneous medium, the second term on the LHS vanishes and each Cartesian component of the electric vector satisfies the scalar wave equation.

In a similar manner taking the curl of Eq. (11.129) and using Eqs. (11.128) and (11.131) we get

$$\nabla^2 \mathscr{H} + \frac{1}{n^2} \nabla n^2 \times (\nabla \times \mathscr{H}) - \epsilon_0 \mu_0 n^2 \frac{\partial^2 \mathscr{H}}{\partial t^2} = 0 \tag{11.137}$$

If the refractive index varies only in the transverse direction, i.e.,

$$n^2 = n^2(x, y) \tag{11.138}$$

then writing each Cartesian component of Eqs. (11.136) and (11.137) one can easily see that the time and z part can be separated out. Thus, if the refractive index is independent of the z-coordinate then the solutions of Eqs. (11.136) and (11.137) can be written in the form

$$\mathscr{E} = \mathbf{E}(x, y)\, e^{i(\omega t - \beta z)} \tag{11.139}$$

$$\mathscr{H} = \mathbf{H}(x, y)\, e^{i(\omega t - \beta z)} \tag{11.140}$$

where β is known as the propagation constant. Eqs. (11.139) and (11.140) define *modes* of the system and were discussed in Sec. 11.2.

We next assume that the refractive index depends only on the x-coordinate, i.e.,

$$n^2 = n^2(x) \tag{11.141}$$

Then even the y part can be separated out implying that the y and z dependences of the fields will be of the form $e^{-i(\gamma y + \beta z)}$. However, we can *always* choose the z-axis along the direction of propagation of the wave and we may, without any loss of generality put $\gamma = 0$. Thus we may write

$$\mathscr{E}_j = E_j(x)\, e^{i(\omega t - \beta z)}; \quad j = x, y, z \tag{11.142}$$

$$\mathscr{H}_j = H_j(x)\, e^{i(\omega t - \beta z)}; \quad j = x, y, z \tag{11.143}$$

Substituting the above expressions for the electric and magnetic fields in Eqs. (11.128) and (11.129) and taking their x, y and z-components we obtain

TE modes:

$$i\beta E_y = -i\omega\mu_0 H_x \tag{11.144}$$

$$\partial E_y/\partial x = -i\omega\mu_0 H_z \tag{11.145}$$

$$-i\beta H_x - \partial H_z/\partial x = i\omega\epsilon_0 n^2(x) E_y \tag{11.146}$$

TM modes:

$$i\beta H_y = i\omega\epsilon_0 n^2(x) E_x \tag{11.147}$$

$$\partial H_y/\partial x = i\omega\epsilon_0 n^2(x) E_z \tag{11.148}$$

$$-i\beta E_x - \partial E_z/\partial x = -i\omega\mu_0 H_y \tag{11.149}$$

These equations were the starting point of Sec. 11.2.

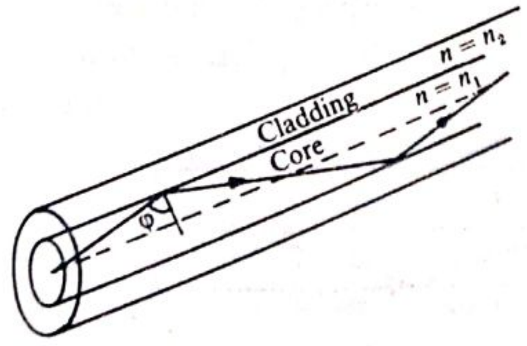
11

Electromagnetic analysis of the simplest optical waveguide

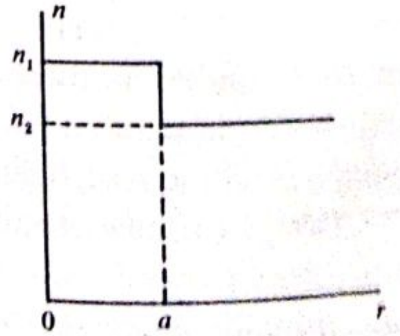
11.1 Introduction

An optical waveguide is a structure which confines and guides the light beam by the process of total internal reflection. The most extensively used optical waveguide is the step index optical fibre which consists of a cylindrical central core, clad by a material of slightly lower refractive index (see Fig. 11.1). If the refractive indices of the core and cladding are n_1 and n_2

Fig. 11.1 (*a*) A typical optical fibre waveguide consists of a thin cylindrical glass rod of radius a and refractive index n_1 clad by glass of slightly lower refractive index n_2. Light guidance takes place through the phenomenon of total internal reflection at the core–cladding interfaces. (*b*) The corresponding refractive index variation; for a typical (multimode) optical fibre $n_1 \approx 1.50$, $n_2 \approx 1.49$, $a \approx 25\ \mu$m.



(*a*)



(*b*)

respectively, then for a ray entering the fibre, if the angle of incidence (at the core–cladding interface) ϕ is greater than the critical angle

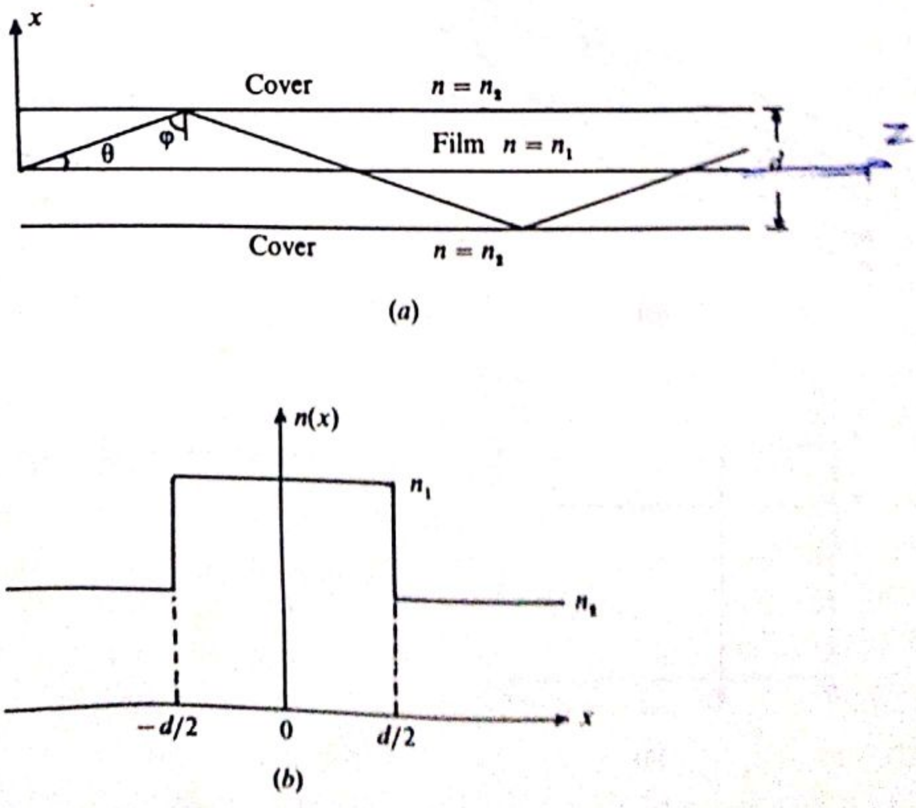
$$\phi_c = \sin^{-1}(n_2/n_1), \tag{11.1}$$

then the ray will undergo total internal reflection at that interface. Further, because of the cylindrical symmetry in the fibre structure, this ray will suffer total internal reflection at the lower interface also and will therefore be guided through the core by repeated total internal reflections. This is the basic principle of light guidance through the optical fibre. We will present a detailed electromagnetic analysis of the waveguiding action in optical fibres in Chapter 13.

The simplest optical waveguide to analyse is probably the planar waveguide which consists of a thin dielectric film (of refractive index n_1) sandwiched between materials of slightly lower refractive indices. Such planar waveguides are important components in integrated optics which will be discussed in Chapter 14.

In this chapter we will present a detailed electromagnetic analysis of the symmetric planar waveguide for which the refractive indices of the materials on the top and bottom of the film are assumed to be the same. Although

Fig. 11.2 (*a*) The simplest planar optical waveguide consists of a planar film (of refractive index n_1) sandwiched between two materials of lower refractive indices. Light guidance takes place by the phenomenon of total internal reflection. (*b*) The refractive index distribution for a symmetric planar waveguide.



almost all waveguides used in integrated optics are asymmetric in nature, we felt that the electromagnetic analysis of a symmetric waveguide is much easier to understand and at the same time it brings out almost all the salient points associated with the modes of a waveguide, therefore making it easier to understand the physical principles of more complicated guiding structures.

In Fig. 11.2 we show the light guidance through a symmetric planar waveguide. The film is assumed to extend to infinity in the y-direction. The propagation is assumed to be in the z-direction. We consider a bundle of rays launched in the film of the waveguide. If ϕ represents the angle that a ray makes with the x-axis then for total internal reflection to occur at the interface of the film and the cover we must have

$$n_2/n_1 < \sin\phi < 1. \tag{11.2}$$

Thus if θ is the angle that the ray makes with the z-axis (see Fig. 11.2), then for total internal reflection to occur (or for light guidance to take place) we must have

$$n_2/n_1 < \cos\theta < 1 \tag{11.3}$$

On the other hand, when

$$\cos\theta < n_2/n_1 \tag{11.4}$$

the angle of incidence at the film–cover interface will be less than the critical angle and the beam will be partially reflected and partially transmitted. After undergoing several such partial reflections, the beam will 'leak away' from the waveguide.

The above considerations are valid in the geometric optics approximation. In the following sections we will give rigorous solutions of Maxwell's equations for the refractive index profile shown in Fig. 11.2 and discuss the concept of guided modes of the waveguide and their relationship to rays. In particular we will show that for the refractive index distribution depending only on the x-coordinate, i.e., for

$$n^2 = n^2(x) \tag{11.5}$$

Maxwell's equations reduce to two independent set of equations: the first set corresponding to what are known as TE (transverse electric) modes where the electric field does not have a longitudinal component and the second set corresponding to what are known as TM (transverse magnetic) modes where the magnetic field does not have a longitudinal component. In particular, for a symmetric waveguide (like the one shown in Fig. 11.2) for which

$$n^2(-x) = n^2(x) \tag{11.6}$$

both TE and TM modes can always be classified under two categories: one symmetric in x and the other antisymmetric in x. We will derive equations from which the propagation characteristics are determined, and will have a detailed discussion on the qualitative characteristics of modes which will be valid for all waveguiding structures.

In summary, this chapter is completely devoted to the quantitative understanding of a simple optical waveguide – but concepts derived from this will be building blocks for more complicated structures.

11.2 Classification of modes for a planar waveguide

In this section we will discuss the broad classification of modes for a planar waveguide – this will be used in the next two sections where we will have a detailed modal analysis for a specific profile.

In Sec. 11.9 we will show that when the refractive index depends only on the x-coordinate, the electric and magnetic fields associated with a propagating electromagnetic wave can be written in the form

$$\mathscr{E}_j = E_j(x)\,\mathrm{e}^{\mathrm{i}(\omega t - \beta z)}; \qquad j = x, y, z \tag{11.7}$$

$$\mathscr{H}_j = H_j(x)\,\mathrm{e}^{\mathrm{i}(\omega t - \beta z)}; \qquad j = x, y, z \tag{11.8}$$

where ω represents the angular frequency of the wave and β is known as the propagation constant. Corresponding to a specific value of β, there is a specific field distribution described by $\mathbf{E}(x)$ and $\mathbf{H}(x)$ and for these specific distributions, the nature of the distribution remains unchanged with propagation along the guide; such distributions are referred to as *modes* of the waveguide. A study of the propagation characteristics and the corresponding field distributions of these modes is of primary importance in the design of efficient integrated optic devices.

If we substitute the above forms of the electric and magnetic fields in Maxwell's equations (Eqs. (11.128) and (11.129)), it can easily be shown that the different components of the electric and magnetic fields are related through the following equations (see Sec. 11.9):

$$\left.\begin{aligned} H_x &= -\frac{\beta}{\omega\mu_0}E_y \qquad &(11.9)\\ H_z &= \frac{\mathrm{i}}{\omega\mu_0}\frac{\partial E_y}{\partial x} \qquad &(11.10)\\ -\mathrm{i}\beta H_x - \frac{\partial H_z}{\partial x} &= \mathrm{i}\omega\epsilon_0 K(x) E_y \qquad &(11.11)\end{aligned}\right\}\ \text{TE modes}$$

$$E_x = \frac{\beta}{\omega\epsilon_0 K(x)} H_y \tag{11.12}$$

$$E_z = \frac{1}{\mathrm{i}\omega\epsilon_0 K(x)} \frac{\partial H_y}{\partial x} \tag{11.13}$$

$$\mathrm{i}\beta E_x + \frac{\partial E_z}{\partial x} = \mathrm{i}\omega\mu_0 H_y \tag{11.14}$$

TM modes

where $K(x) = n^2(x)$, ϵ_0 and μ_0 are the dielectric permittivity and magnetic permeability of free space. As can be seen, the first three equations involve only E_y, H_x and H_z and the last three equations involve only E_x, E_z and H_y. Thus for such a waveguide configuration, Maxwell's equations reduce to two independent sets of equations. The first set corresponds to nonvanishing values of E_y, H_x and H_z and with E_x, E_z and H_y vanishing, giving rise to what are known as transverse electric (TE) modes because the electric field has only a transverse component. The second set corresponds to the nonvanishing values of E_x, E_z and H_y with E_y, H_x and H_z vanishing, giving rise to what are known as transverse magnetic (TM) modes because the magnetic field now has only a transverse component. The propagation of waves in such planar waveguides may thus be described in terms of TE and TM modes. In the next two sections we will discuss the TE and TM modes of a symmetric step index planar waveguide.

11.3 TE modes of a symmetric step index planar waveguide

In this and the following section we will carry out a detailed modal analysis of a symmetric step index planar waveguide. We first consider TE modes: we substitute H_x and H_z from Eqs. (11.9) and (11.10) in Eq. (11.11) to obtain

$$\mathrm{d}^2 E_y/\mathrm{d}x^2 + [k_0^2 n^2(x) - \beta^2]E_y = 0 \tag{11.15}$$

where

$$k_0 = \omega(\epsilon_0\mu_0)^{\frac{1}{2}} = \omega/c \tag{11.16}$$

is the free space wave number and c $(= 1/(\epsilon_0\mu_0)^{\frac{1}{2}})$ is the speed of light in free space.

Until now our analysis has been valid for an arbitrary x dependent profile. We now assume a specific profile given by (see Fig. 11.2)

$$n(x) = \begin{cases} n_1; & |x| < d/2 \\ n_2; & |x| > d/2 \end{cases} \tag{11.17}$$

Using the above equations we will solve Eq. (11.15) subject to the appropriate boundary conditions at the discontinuities. Since E_y and H_z represent tangential components on the planes $x = \pm d/2$, they must be continuous at $x = \pm d/2$ and since H_z is proportional to dE_y/dx (see Eq. (11.10)) we must have

$$E_y \text{ and } dE_y/dx \text{ continuous at } x = \pm d/2 \tag{11.18}$$

The above condition represents the boundary conditions that have to be satisfied.† Substituting for $n(x)$ in Eq. (11.15) we obtain

$$d^2E_y/dx^2 + (k_0^2 n_1^2 - \beta^2)E_y = 0; \quad |x| < d/2 \text{ film} \tag{11.19}$$

$$d^2E_y/dx^2 + (k_0^2 n_2^2 - \beta^2)E_y = 0; \quad |x| > d/2 \text{ cover} \tag{11.20}$$

For guided modes we require that the field should decay in the cover (i.e., in the region $|x| > d/2$) so that most of the energy associated with the mode lies inside the film. Thus we must have‡

$$\beta^2 > k_0^2 n_2^2 \tag{11.21}$$

Furthermore, we must also have $\beta^2 < k_0^2 n_1^2$, otherwise the boundary conditions cannot be satisfied§ at $x = \pm d/2$. Thus for guided modes we must have

$$k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2 \tag{11.22}$$

We therefore write Eqs. (11.19) and (11.20) in the form

$$d^2E_y/dx^2 + \kappa^2 E_y = 0; \quad |x| < d/2 \text{ film} \tag{11.23}$$

$$d^2E_y/dx^2 - \gamma^2 E_y = 0; \quad |x| > d/2 \text{ cover} \tag{11.24}$$

where

$$\kappa^2 = k_0^2 n_1^2 - \beta^2 \tag{11.25}$$

† The very fact that E_y satisfies Eq. (11.15) also implies that E_y and dE_y/dx are continuous unless $n^2(x)$ has an infinite discontinuity. This follows from the fact that if E_y' is discontinuous then E_y'' will be a delta function and Eq. (11.15) will lead to an inconsistent equation. Thus the continuity conditions are imbedded in Maxwell's equations.

‡ When $\beta^2 < k_0^2 n_2^2$, the solutions are oscillatory in the region $|x| > d/2$ and they correspond to what are known as radiation modes of the waveguide. These modes correspond to rays which undergo refraction (rather than total internal reflection) at the film–cover interface and when these are excited, they quickly leak away from the core of the waveguide. Some aspects of radiation modes will be discussed in Chapter 12.

§ It is left as an exercise for the reader to show that if we assume $\beta^2 > k_0^2 n_1^2$ and also assume decaying fields in the region $|x| > d/2$ then the boundary conditions at $x = +d/2$ *and* at $x = -d/2$ can *never* be satisfied (see also Problem 11.8).

and

$$\gamma^2 = \beta^2 - k_0^2 n_2^2 \tag{11.26}$$

The solution of Eq. (11.23) can be written in the form

$$E_y(x) = A\cos\kappa x + B\sin\kappa x; \qquad |x| < d/2 \tag{11.27}$$

where A and B are constants. In the region $x > d/2$ and $x < -d/2$ the solutions are $e^{\pm\gamma x}$ and if we neglect the exponentially amplifying one, we will obtain

$$E_y(x) = \begin{cases} Ce^{\gamma x}; & x < -d/2 \\ De^{-\gamma x}; & x > d/2 \end{cases} \tag{11.28}$$

If we now apply the boundary conditions (*viz.*, continuity of E_y and dE_y/dx at $x = \pm d/2$) we will get four equations from which we can get the transcendental equation, which will determine the allowed values of β. This is indeed the general procedure for determining the propagation constants (see e.g., Sec. 14.2); however, when the refractive index distribution is symmetric about $x = 0$, i.e., when

$$n^2(-x) = n^2(x) \tag{11.29}$$

the solutions are either symmetric or antisymmetric functions of x; thus we must have

$$E_y(-x) = E_y(x) \quad \text{symmetric modes} \tag{11.30}$$

$$E_y(-x) = -E_y(x) \quad \text{antisymmetric modes} \tag{11.31}$$

(The proof of this theorem is discussed in Problem 11.10.) For the symmetric mode, we must have

$$E_y(x) = \begin{cases} A\cos\kappa x; & |x| < d/2 \quad (11.32) \\ Ce^{-\gamma|x|}; & |x| > d/2 \quad (11.33) \end{cases}$$

Continuity of $E_y(x)$ and dE_y/dx at $x = \pm d/2$ gives us

$$A\cos(\kappa d/2) = Ce^{-\gamma d/2} \tag{11.34}$$

and

$$-\kappa A\sin(\kappa d/2) = -\gamma Ce^{-\gamma d/2} \tag{11.35}$$

respectively. Dividing Eq. (11.35) by Eq. (11.34) we get

$$(\kappa d/2)\tan(\kappa d/2) = (\gamma d/2) \tag{11.36}$$

Since

$$\gamma^2 = \beta^2 - k_0^2 n_2^2 = k_0^2(n_1^2 - n_2^2) - \kappa^2 \tag{11.37}$$

we may write

$$\gamma d/2 = (V^2/4 - \xi^2)^{\frac{1}{2}} \tag{11.38}$$

where

$$\xi = \kappa d/2 = (k_0^2 n_1^2 - \beta^2)^{\frac{1}{2}} d/2 \tag{11.39}$$

and

$$V = k_0 d(n_1^2 - n_2^2)^{\frac{1}{2}} \tag{11.40}$$

is known as the dimensionless waveguide parameter. Thus Eq. (11.36) can be put in the following form:

$$\xi \tan \xi = (V^2/4 - \xi^2)^{\frac{1}{2}} \tag{11.41}$$

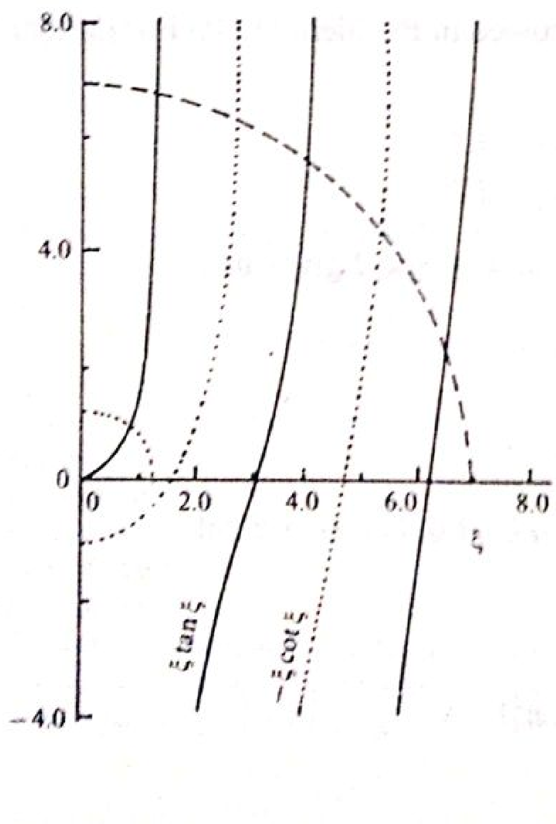
Similarly, for the antisymmetric mode we will have

$$E_y(x) = \begin{cases} B \sin \kappa x; & |x| < d/2 \quad (11.42) \\ \dfrac{x}{|x|} D e^{-\gamma |x|}; & |x| > d/2 \quad (11.43) \end{cases}$$

Following an exactly similar procedure we get

$$-\xi \cot \xi = (V^2/4 - \xi^2)^{\frac{1}{2}} \tag{11.44}$$

Fig. 11.3 The variation of $\xi \tan \xi$ (solid curves) and $-\xi \cot \xi$ (dotted curves) as a function of ξ. The points of intersection of the solid and dotted curves with the quadrant of a circle of radius $V_0(=V/2)$ determine the propagation constants of the optical waveguide corresponding to symmetric and antisymmetric modes respectively.



Thus, we have

$$\xi \tan \xi = [V_0^2 - \xi^2]^{\frac{1}{2}} \quad \text{symmetric case} \tag{11.45}$$

$$-\xi \cot \xi = [V_0^2 - \xi^2]^{\frac{1}{2}} \quad \text{antisymmetric case} \tag{11.46}$$

where

$$V_0 = V/2 = k_0 d(n_1^2 - n_2^2)^{\frac{1}{2}}/2 \tag{11.47}$$

Since the equation

$$\eta = (V_0^2 - \xi^2)^{\frac{1}{2}} \tag{11.48}$$

represents a portion of a circle (of radius V_0) in the (ξ, η) plane, the numerical evaluation of the allowed values of ξ (and hence of the propagation constants) is quite simple. In Fig. 11.3 we have plotted the functions $\xi \tan \xi$ (solid curve) and $-\xi \cot \xi$ (dotted curve) as a function of ξ. Their points of intersection with the quadrant of the circle determine the allowed values of ξ and if we use Eq. (11.39) we can determine the corresponding values of β.

11.3.1 *Some general comments about the modes*

From Fig. 11.3 we can derive the following conclusions:

(*a*) If $0 < V_0 < \pi/2$, i.e., when

$$0 < V < \pi \tag{11.49}$$

we have only one discrete (TE) mode of the waveguide and this mode is symmetric in x. When this happens, we refer to the waveguide as a 'single moded waveguide'. For example, if $\lambda_0 \approx 1.5\ \mu\text{m}$, $n_1 = 1.50$, $n_2 = 1.48$ then for single mode operation we must have

$$\frac{2\pi}{1.5} d[(1.50)^2 - (1.48)^2]^{\frac{1}{2}} < \pi$$

where d is measured in microns. Solving we get

$$d < 3.07\ \mu\text{m}$$

We will show below that if the operating wavelength is made smaller, the same waveguide will be able to support more than one mode.

(*b*) From Fig. 11.3 it is easy to see that if $\pi/2 < V_0 < \pi$ (or, $\pi < V < 2\pi$) we will have one symmetric and one antisymmetric mode. In general, if

$$2m\pi < V < (2m+1)\pi \tag{11.50}$$

we will have $(m+1)$ symmetric modes, and m antisymmetric modes and if

$$(2m+1)\pi < V < (2m+2)\pi \tag{11.51}$$

we will have $(m+1)$ symmetric modes, and $(m+1)$ antisymmetric modes where $m = 0, 1, 2, \ldots$. Thus the total number of modes will be the integer closest to (and greater than) V/π. Thus for the waveguide considered above, if the operating wavelength is made 0.6 μm then $V = 2.5\pi$ and therefore we will have three modes (two symmetric and one antisymmetric).

(*c*) When the waveguide supports many modes (i.e., when $V \gg 1$) the points of intersection (in Fig. 11.3) will be very close to $\xi = \pi/2, \pi, 3\pi/2$, etc; thus the propagation constants corresponding to the first few modes will be approximately given by the following equation

$$\xi = \xi_m = (k_0^2 n_1^2 - \beta_m^2)^{\frac{1}{2}} d/2 \approx (m+1)\pi/2; \qquad V \ggg 1 \tag{11.52}$$

where

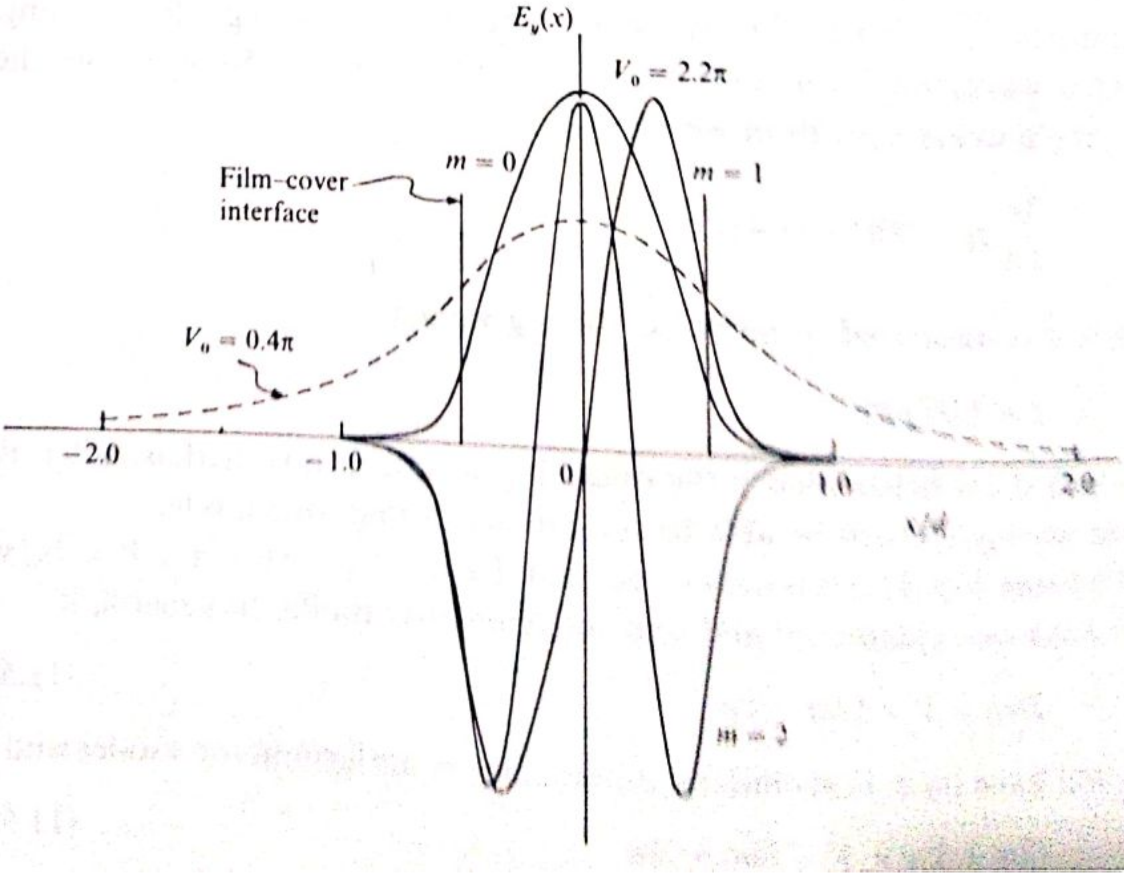
$m = 0, 2, 4, \ldots$ correspond to symmetric modes

and

$m = 1, 3, 5, \ldots$ correspond to antisymmetric modes

(*d*) It is obvious from Fig. 11.3 that for the fundamental mode (which we will refer to as the zero order mode), $\xi\ (= \kappa d/2)$ will *always* lie between 0 and

Fig. 11.4 The solid curves represent the modal fields for the symmetric step index planar waveguide with $V = 4.4\pi$; even and odd values of m correspond to symmetric and antisymmetric modes respectively. The dashed curve represents the fundamental mode for $V = 0.8\pi$. All modes have been normalized to carry the same power.



$\pi/2$ and the corresponding field variation $E_y(x)$ will have no zeroes. For the next mode (which will be antisymmetric in x) ξ ($=\kappa d/2$) will always lie between $\pi/2$ and π and therefore the corresponding $E_y(x)$ will have only one zero (at $x = 0$). It is easy to extend the analysis and prove that

the m^{th} mode will have m zeroes (11.53)

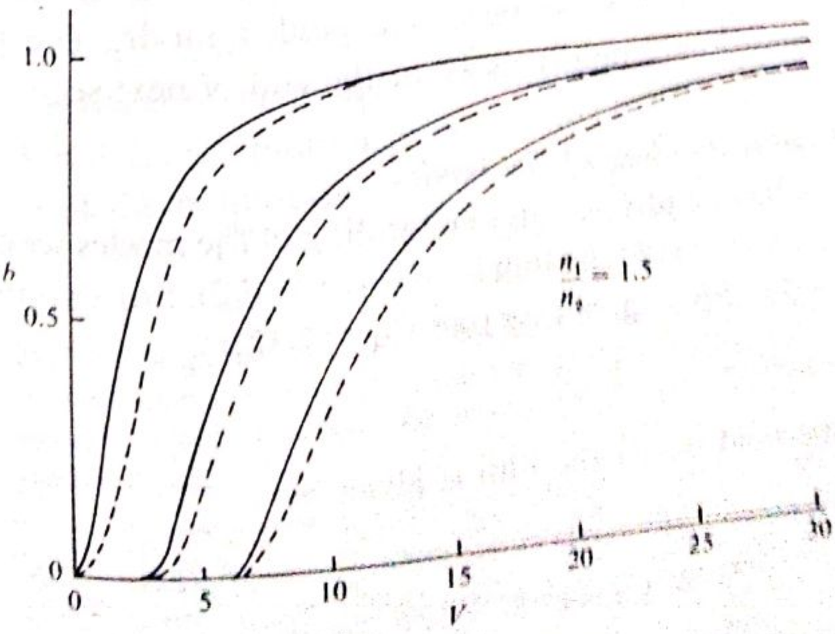
The above statement is valid for an arbitrary waveguiding structure. The actual plot of the modal pattern for the first few modes is shown in Fig. 11.4. It may be noted that the field spreads out more as the wavelength increases or V number decreases.

(e) We define a dimensionless parameter

$$b \equiv \frac{\beta^2/k_0^2 - n_2^2}{n_1^2 - n_2^2} = 1 - \frac{\xi^2}{V^2/4} \tag{11.54}$$

(The quantity b is usually referred to as the normalized propagation constant). The allowed values of ξ (and hence of b) are calculated using Eqs. (11.45) and (11.46) for different values of V. The corresponding variations of b with V are plotted in Fig. 11.5 for the first few modes. The curves are universal, i.e., for a given waveguide and a given operating wavelength we have first to determine V and then to 'read off' from the curves the exact values of b from which we can determine the values of β by using the

Fig. 11.5 The variation of the normalized propagation constant b as a function of V for a symmetric step index planar waveguide. The solid and the dashed curves correspond to TE and TM modes respectively. The value of V at which $b = 0$ corresponds to what is known as the cutoff frequency.



following equation:

$$\beta^2 = k_0^2[n_2^2 + b(n_1^2 - n_2^2)] \tag{11.55}$$

It may be noted that since for guided modes

$$n_2^2 < \beta^2/k_0^2 < n_1^2 \tag{11.56}$$

We must have

$$0 < b < 1 \tag{11.57}$$

Since for a guided mode β cannot be less than n_2k_0, when (for a particular mode) β reaches the value equal to n_2k_0 (i.e., when b becomes equal to zero), the mode is said to have reached 'cutoff'. Thus at cutoff

$$\beta = n_2k_0, \qquad \gamma = 0 \qquad \text{and} \qquad b = 0$$

For the symmetric waveguide that we have been discussing, at cutoff $\xi = V/2 = V_0$ and hence the cutoff of TE modes is determined by

$$\frac{V}{2}\tan\left(\frac{V}{2}\right) = 0 \qquad \text{symmetric modes}$$

$$\frac{V}{2}\cot\left(\frac{V}{2}\right) = 0 \qquad \text{antisymmetric modes}$$

The above equation implies that the cutoff V values for various modes are given by

$$V_c = m\pi; \qquad m = 0, 1, 2, 3, \ldots \tag{11.58}$$

where even and odd values of m correspond to symmetric and antisymmetric modes respectively. Notice that the fundamental mode has no cutoff and therefore there will always be at least one guided mode. The physical understanding of cutoff will be discussed at the end of next section.

11.3.2 *Physical understanding of the modes*

In order to have a physical understanding of the modes we consider the electric field pattern inside the film ($-d/2 < x < d/2$). For example, for a symmetric TE mode, this is given by (see Eq. (11.32)):

$$E_y(x) = A\cos\kappa x$$

Thus the complete field inside the film is given by

$$\begin{aligned}\mathscr{E}_y &= A\cos\kappa x\, e^{i(\omega t - \beta z)} \\ &= \tfrac{1}{2}A\, e^{i(\omega t - \beta z - \kappa x)} + \tfrac{1}{2}A\, e^{i(\omega t - \beta z + \kappa x)}\end{aligned} \tag{11.59}$$

Now

$$e^{i(\omega t - k_x x - k_y y - k_z z)} \tag{11.60}$$

represents a wave propagating along the direction of **k** whose x, y and z-components are k_x, k_y and k_z respectively. Thus for the two terms on the RHS of Eq. (11.59) we have

$$k_x = \pm \kappa, k_y = 0 \quad \text{and} \quad k_z = \beta \tag{11.61}$$

which represent plane waves with propagation vectors parallel to the x–z plane making angles $\pm\theta$ with the z-axis where

$$\tan\theta = k_x/k_z = \kappa/\beta \tag{11.62}$$

or

$$\cos\theta = \beta/(\beta^2 + \kappa^2)^{\frac{1}{2}} = \beta/k_0 n_1 \tag{11.63}$$

Thus a guided mode can be considered to be a superposition of a pair of plane waves which are propagating at angles $\pm\theta\,(=\pm\cos^{-1}(\beta/k_0 n_1))$ with the z-axis (see Fig. 11.6). Since only discrete values of β are allowed (which we designate as β_m), only discrete angles of propagation of waves (or of the rays) are allowed. Each mode is characterised by a *discrete* angle of propagation θ_m. We will use this concept to derive the eigenvalue equation in Sec. 14.3.

The concept of the cutoff of a mode can also be easily understood from the above discussion. Since the guided waves correspond to

$$n_2 < \beta/k_0 < n_1 \tag{11.64}$$

we have

$$n_2/n_1 < \cos\theta < 1 \tag{11.65}$$

The condition that β cannot be less than $n_2 k_0$ implies that $\cos\theta$ should be greater than n_2/n_1 which is nothing but the condition for total internal reflection at the core–cladding interface (see Eq. (11.3)). Thus beyond cutoff, i.e., for $V < V_c$, the component waves no longer undergo total internal reflections at the boundaries.

Fig. 11.6 A guided mode in a step index waveguide corresponds to the superposition of two plane waves (inside the film) propagating at particular angles $\pm\theta$ with the z-axis. Different modes will correspond to different (discrete) values of θ.

11.4 TM modes of a symmetric step index planar waveguide

In the above discussion we have considered the TE modes of the waveguide. An exactly similar analysis can also be performed for the TM modes which are characterized by field components E_x, E_z and H_y (see Eqs. (11.12)–(11.14)). If we substitute for E_x and E_z from Eqs. (11.12) and (11.13) in Eq. (11.14) we will get

$$n^2(x)\frac{d}{dx}\left[\frac{1}{n^2(x)}\frac{dH_y}{dx}\right]+(k_0^2n^2-\beta^2)H_y=0 \tag{11.66}$$

which can be rewritten as

$$\frac{d^2H_y}{dx^2}-\left[\frac{1}{n^2(x)}\frac{dn^2}{dx}\right]\frac{dH_y}{dx}+[k_0^2n^2(x)-\beta^2]H_y(x)=0 \tag{11.67}$$

The above equation is of a form which is somewhat different from the equation satisfied by E_y for TE modes (see Eq. (11.15)); however, for the step index waveguide shown in Fig. 11.2, the refractive index is constant in each region and therefore we have

$$d^2H_y/dx^2+(k_0^2n_1^2-\beta^2)H_y(x)=0; \quad |x|<d/2 \tag{11.68}$$

and

$$d^2H_y/dx^2-(\beta^2-k_0^2n_2^2)H_y(x)=0; \quad |x|>d/2 \tag{11.69}$$

We must be careful about the boundary conditions. Since H_y and E_z are tangential components on the planes $x=\pm d/2$, we must have (see Eq. (11.13))

$$H_y \text{ and } \frac{1}{n^2}\frac{dH_y}{dx} \text{ continuous at } x=\pm\frac{d}{2} \tag{11.70}$$

This is also obvious from Eq. (11.66)†. The solutions of Eqs. (11.68) and (11.69) can be written immediately. Considering first the symmetric modes, we have

$$H_y(x)=\begin{cases} A\cos\kappa x; & |x|<d/2 \quad (11.71)\\ B\,e^{-\gamma|x|}; & |x|>d/2 \quad (11.72)\end{cases}$$

where the symbols κ and γ are the same as given by Eqs. (11.25) and (11.26).

† Once again the condition that H_y and $(1/n^2)\,dH_y/dx$ should be continuous at $x=\pm d/2$ follows from Eq. (11.66) because if $(1/n^2)\,dH_y/dx$ was discontinuous $d/dx[(1/n^2)H_y']$ would be a delta function and Eq. (11.66) would lead to an inconsistent equation. Thus the continuity of H_y and $(1/n^2)H_y'$ are contained in Eq. (11.66).

The boundary conditions given by Eq. (11.70) give us

$$A\cos(\kappa d/2) = B\,e^{-\gamma d/2} \tag{11.73}$$

$$\frac{1}{n_1^2}\left(-A\kappa\sin\frac{\kappa d}{2}\right) = \frac{1}{n_2^2}\left(-B\gamma e^{-\gamma d/2}\right) \tag{11.74}$$

Dividing we get

$$\kappa\tan(\kappa d/2) = (n_1^2/n_2^2)\gamma \tag{11.75}$$

which can be rewritten in the form

$$\xi\tan\xi = (n_1^2/n_2^2)(V_0^2-\xi^2)^{\frac{1}{2}} \qquad \text{symmetric TM modes} \tag{11.76}$$

A similar derivation gives us

$$-\xi\cot\xi = (n_1^2/n_2^2)(V_0^2-\xi^2)^{\frac{1}{2}} \qquad \text{antisymmetric TM modes} \tag{11.77}$$

where, as before,

$$\xi = \kappa d/2 = (k_0^2 n_1^2 - \beta^2)^{\frac{1}{2}} d/2$$

and

$$V_0 = V/2 = k_0 d(n_1^2 - n_2^2)^{\frac{1}{2}}/2$$

The numerical solutions of Eqs. (11.76) and (11.77) can be discussed in a manner exactly similar to the TE case with the difference that the RHS of Eqs. (11.76) and (11.77) now represents an ellipse whose semimajor axis (along the η-direction) is of magnitude $(n_1/n_2)V_0$ and whose semi-minor axis (along the ξ-direction) is of magnitude V_0. All qualitative conclusions discussed in Sec. 11.3 for TE modes (*viz.*, the cutoff frequencies and number of zeroes of various modes, the physical interpretation of modal fields etc.) will remain valid. We should also mention the following three points:

(*a*) Although a waveguide for which $0 < V < \pi$ is referred to as a single moded waveguide, we actually have two modes (one TE and one TM) characterized by slightly different propagation constants. However, the incident field is usually linearly polarized and if **E** is along the y-axis, the TE mode is excited and if **E** is along the x-axis, the TM mode is excited. *This result is quite general and is valid for all planar waveguides*. On the other hand, if the incident field has a polarization which makes an angle with the x-axis (or, if the field is elliptically polarized) then both TE and TM modes will be excited and because they have slightly different propagation constants, they will superpose with different phases at different values of z changing the state of the resultant polarization. As an example, we consider the incidence of a linearly polarized wave with the electric vector making an angle of 45° with the x and y-axes. Thus at $z = 0$ we have

$$\left.\begin{aligned}\mathscr{E}_x &= \mathscr{E}_0 \cos 45 \cos \omega t\\ \mathscr{E}_y &= \mathscr{E}_0 \sin 45 \cos \omega t\end{aligned}\right\} \begin{array}{l}\text{Field distribution}\\ \text{at } z = 0\end{array} \qquad (11.78)$$

where $\mathscr{E}_0$ represents the transverse variation of the modal field which we have assumed to be the same for the TE and TM modes (see Eqs. (11.32) and (11.71) – the values of β are assumed to be nearly equal). If the propagation constants for the TE and TM modes are denoted by β_0 and $(\beta_0 - \Delta\beta_0)$ respectively, then the field distributions for $z > 0$ will be

$$\left.\begin{aligned}\text{TE:}\quad \mathscr{E}_y &= \frac{\mathscr{E}_0}{\sqrt{2}} \cos[\omega t - \beta_0 z]\\ \text{TM:}\quad \mathscr{E}_x &= \frac{\mathscr{E}_0}{\sqrt{2}} \cos[\omega t - \beta_0 z + \Delta\beta_0 z]\end{aligned}\right\} \begin{array}{l}\text{Field distribution}\\ \text{at } z > 0\end{array} \qquad (11.79)$$

It can be readily seen that at $z = \pi/(2\Delta\beta_0)$ the beam will be circularly polarized and at $z = \pi/(\Delta\beta_0)$ the beam will be linearly polarized (with the electric vector now at right angles to the original direction) – for intermediate values of z, the beam will be elliptically polarized. For a distance $z = L_b = 2\pi/\Delta\beta_0$, the original polarization state is restored and this characteristic length is referred to as the beat length.

(*b*) Similarly, for $\pi < V < 2\pi$, although the waveguide is referred to as a 'two-moded waveguide' there are actually four modes (two TE and two TM) etc.

(*c*) For most practical waveguides, $n_1 \approx n_2$ and the propagation constants (and the field patterns) for the TE and TM modes are very nearly equal.

Problem 11.1: Consider a planar symmetric waveguide with $n_1 = 1.5$, $n_2 = 1.0$ and $V = 3.0$. Assuming $\lambda_0 = 1.3\ \mu$m calculate $\Delta\beta_0$ and obtain the corresponding beat length.
(Answer: $\beta_{TE} = 6.4574\ \mu m^{-1}$, $\beta_{TM} = 6.0393\ \mu m^{-1}$, $L_b \approx 15\ \mu m$.)

11.5 The relative magnitude of the longitudinal components of the *E* and *H* fields

We first consider the TE modes. Using Eqs. (11.9) and (11.10) we get

$$\left|\frac{H_z}{H_x}\right| = \frac{1}{\beta}\left|\frac{\partial E_y/\partial x}{E_y}\right| \qquad (11.80)$$

Now from Eq. (11.32) we have (inside the film)

$$\left|\frac{E_y}{\partial E_y/\partial x}\right| \sim \frac{1}{\kappa} \qquad (11.81)$$

which represents the characteristic distance for the spatial variation in the x-direction. Thus

$$|H_z/H_x| \sim \kappa/\beta \tag{11.82}$$

Since

$$\kappa^2 = k_0^2 n_1^2 - \beta^2$$

we readily have

$$\kappa/\beta = (k_0^2 n_1^2/\beta^2 - 1)^{\frac{1}{2}} \tag{11.83}$$

Now, the guided modes correspond to

$$n_2^2 < \beta^2/k_0^2 < n_1^2$$

therefore

$$0 < \frac{\kappa}{\beta} < \left(\frac{n_1^2 - n_2^2}{n_2^2}\right)^{\frac{1}{2}} \tag{11.84}$$

Thus

$$\left|\frac{H_z}{H_x}\right| \lesssim \left(\frac{n_1^2 - n_2^2}{n_2^2}\right)^{\frac{1}{2}} \tag{11.85}$$

For $n_1 \approx 1.50$ and $n_2 \approx 1.49$, the RHS of the above equation is about 0.1 which shows that the longitudinal component is very weak in comparison to the transverse component. Thus, as long as $n_1 \approx n_2$, the mode can be approximately assumed to be a 'transverse electromagnetic mode'. The same is also true for the TM modes.

The above discussion is valid, in general, as long as $n_1 \approx n_2$. In Sec. 14.2 we will discuss asymmetric waveguides where the refractive index of the material on top of the film (n_c) is different from that of the material below the film (n_s). However, as long as n_c and n_s have values which are close to the refractive index of the film, the mode will be approximately transverse in both **E** and **H** fields. In Sec. 13.5 we will discuss round optical waveguides with cylindrical symmetry – even there, as long as the core and cladding refractive indices are nearly equal (which is true for *all* practical fibres) the longitudinal components of the electric and magnetic fields are usually negligible in comparison to the corresponding transverse components. When this is the case, the waveguide is referred to as '*weakly guiding*'.

11.6 Power associated with a mode

In this section we will calculate the power associated with the TE mode. The power flow is given by (see Eq. (1.92))

$$\langle \mathbf{S} \rangle = \tfrac{1}{2}\,\mathrm{Re}\,(\mathcal{E} \times \mathcal{H}^*) \tag{11.86}$$

Now for the TE mode we have (see Eqs. (11.7)–(11.11))

$$\mathscr{E}_y = E_y(x)\,e^{i(\omega t - \beta z)} \tag{11.87}$$

$$\mathscr{H}_x = -\frac{\beta}{\omega\mu_0}\mathscr{E}_y = -\frac{\beta}{\omega\mu_0}E_y(x)\,e^{i(\omega t - \beta z)} \tag{11.88}$$

and

$$\mathscr{H}_z = \frac{i}{\omega\mu_0}\frac{\partial \mathscr{E}_y}{\partial x} = \frac{i}{\omega\mu_0}\frac{dE_y}{dx}e^{i(\omega t - \beta z)} \tag{11.89}$$

Using the above equations we readily get $\langle S_y \rangle = 0$ and

$$\langle S_x \rangle = \tfrac{1}{2}\mathrm{Re}(\mathscr{E}_y \mathscr{H}_z^*) = 0 \tag{11.90}$$

Since E_y is real. Further,

$$\langle S_z \rangle = -\tfrac{1}{2}\mathrm{Re}\langle \mathscr{E}_y \mathscr{H}_x^* \rangle \tag{11.91}$$

or

$$\langle S_z \rangle = \frac{\beta}{2\omega\mu_0}|E_y|^2 \tag{11.92}$$

Although the above expression is rigorously valid only for the TE mode in a slab waveguide, it is approximately valid for all waveguides in the weakly guiding approximation.

The power associated with the mode (per unit length in the y-direction) is given by

$$P = \frac{1}{2}\frac{\beta}{\omega\mu_0}\int_{-\infty}^{+\infty}|E_y|^2\,dx \tag{11.93}$$

We consider the symmetric mode (see Eqs. (11.32) and (11.33)) for which

$$P = \frac{1}{2}\frac{\beta}{\omega\mu_0}2\left(A^2\int_0^{d/2}\cos^2\kappa x\,dx + C^2\int_{d/2}^{\infty}e^{-2\gamma x}\,dx\right) \tag{11.94}$$

or

$$P = \frac{\beta}{2\omega\mu_0}A^2\left(\frac{d}{2} + \frac{1}{2\kappa}\sin\kappa d + \frac{C^2}{A^2}\frac{1}{\gamma}e^{-\gamma d}\right)$$

If we now use Eq. (11.34) for C/A we get

$$P = \frac{\beta A^2}{4\omega\mu_0}\left\{d + \frac{2\sin(\kappa d/2)\cos(\kappa d/2)}{\kappa} + \frac{2}{\gamma}[1 - \sin^2(\kappa d/2)]\right\}$$

$$= \frac{\beta A^2}{4\omega\mu_0}\left\{d + \frac{2}{\gamma} + \frac{2\sin(\kappa d/2)\cos(\kappa d/2)}{\gamma\kappa}[\gamma - \kappa\tan(\kappa d/2)]\right\}$$

$$P = \frac{\beta A^2}{4\omega\mu_0}\left(d + \frac{2}{\gamma}\right) \tag{11.95}$$

where we have used Eq. (11.36).

Problem 11.2: Show that even for the antisymmetric TE mode, the power associated (per unit length in the y-direction) is given by Eq. (11.95).

Problem 11.3: Carry out the analysis of power flow of TM modes and show that

$$P = \frac{A^2\beta}{2\omega\epsilon_0 n_1^2}\left[\frac{d}{2} + \frac{(n_1 n_2)^2}{\gamma}\frac{k_0^2(n_1^2 - n_2^2)}{(n_2^4\kappa^2 + n_1^4\gamma^2)}\right] \tag{11.96}$$

for both symmetric as well as antisymmetric modes.

Problem 11.4: Consider a symmetric planar waveguide with the following parameters:

$$n_1 = 1.50, \quad n_2 = 1.48, \quad d = 3.912\,\mu\text{m}$$

At $\lambda_0 = 1\,\mu$m, (*a*) show that there will be only two TE modes, the corresponding propagation constants being $\beta_0 = 9.4058\,\mu\text{m}^{-1}$ and $\beta_1 = 9.3525\,\mu\text{m}^{-1}$. (*b*) At $z = 0$ assume that the field in the core is given by

$$E_y(x) = 1.375 \times 10^4 \cos\kappa_0 x e^{i\omega t} + 1.309 \times 10^4 \sin\kappa_1 x e^{i\omega t}\, V/\text{m}$$

Show that equal power of 1 W is carried by the two modes. Calculate the transverse intensity distribution at

$$z = 0, \qquad \pi/\Delta\beta, \qquad 2\pi/\Delta\beta$$

where $\Delta\beta = \beta_0 - \beta_1$. Interpret the results physically.

11.7 Radiation modes

Till now we have considered the guided modes of the waveguide for which

$$n_2^2 < \beta^2/k_0^2 < n_1^2.$$

There exists another class of modes for which†

$$\beta^2/k_0^2 < n_2^2 \tag{11.97}$$

These are referred to as the *radiation modes* of the waveguide. It can be immediately seen that for $(\beta^2/k_0^2) < n_2^2$ the wave equation (say for the TE modes) in the region $|x| > d/2$ takes the form

$$\mathrm{d}^2E_y/\mathrm{d}x^2 + \delta^2 E_y = 0 \tag{11.98}$$

where

$$\delta^2 = k_0^2 n_2^2 - \beta^2 \tag{11.99}$$

which is now a positive quantity, thus the solutions in the region $|x| > d/2$ will be wavelike of the form

$$e^{\pm i\delta x} \tag{11.100}$$

† It is impossible to have $\beta/k_0 > n_1$ (see Problem 11.8).