① GE-2 Linear Algebra
Section-③ Sem-②

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Linear Transformation

Although a vector can be used to indicate a particular type of movement, actual vectors themselves are essentially static, unchanging objects. For example if we represent the edges of a ~~particular~~ particular image on a computer screen by vectors, then these vectors are fixed in place. However we want to move or alter the image in some way, such as rotating it about a point on the screen, we need a function to calculate the new position for each of the original vectors.

This suggests that we need another "tool" in our arsenal: functions that move a given set of vectors in a prescribed "linear" manner. Such functions are called linear transformations.

Definition:- Let V and W be vector spaces, and let $f: V \to W$ be a function from V to W. Then f is a linear transformation iff the following are true

① $f(v_1+v_2) = f(v_1)+f(v_2) \quad \forall \; v_1, v_2 \in V$

② $f(cv) = cf(v) \; \forall \; v \in V \; \& \; \forall \; c \in R$

properties ① & ② show that the operations of addition and scalar multiplication give the same result on vectors whether the operations are performed before f is applied (in V) or after f is applied (in W).

Thus, a linear transformation is a function between vector spaces that "preserves" the operations that give structure to the spaces.

Ex (1) the mapping $f: M_{mn} \to M_{nm}$ given by $f(A) = A^T$ for any $m \times n$ matrix A, we will show that f is a linear transformation.

Soln:- To show any function is linear transformation we need to show the above two properties of definition are holds. hence

(1) ~~here~~ we need to show that
$$f(A_1 + A_2) = f(A_1) + f(A_2) \cdot \text{ for } A_1, A_2 \in M_{mn}$$

However $f(A_1 + A_2) = (A_1 + A_2)^T = A_1^T + A_2^T$
$(\because (A_1 + A_2)^T = A_1^T + A_2^T)$

$$\boxed{f(A_1 + A_2) = f(A_1) + f(A_2)}$$

(2) we need to show that
$f(cA) = c f(A)$ for any $c \in \mathbb{R}$ and any ~~$A \in M_{mn}$~~ $A \in M_{mn}$

$$f(cA) = (cA)^T = c A^T = c f(A)$$
(because c is real) $c^T = c$ ✓

~~Hence f is a linear transformation~~

Ex (2) Consider $g: P_n \to P_{n-1}$ given $g(P) = P'$ the derivative of P, show that g is linear transformation

~~Ex (2)~~ Consider the mapping $f: V \to \mathbb{R}^n$ given by ~~$f(v) = [v]_B$~~, ~~show that f is linear~~ transformation.

~~Ex (3) Consider the function~~ $h: \mathbb{R}^2 \to \mathbb{R}^2$ ~~given by~~ $h(x, y) = (x+1, y-2)$, verify that given function is linear or not.

Linear operator:- Let V be vector space. A linear operator on V is a linear transformation whose domain and co-domain both are ~~...~~ V.

Hence if $f: V \to V$ be a linear transformation then it is a linear operator also.

Ex:- ~~the mapping~~ $i: V \to V$ given by $i(v) = v$ $\forall v \in V$ is a linear operator, known as the identity linear operator.

* The mapping $Z: V \to V$ given by
$Z(v) = 0 \; \forall \; v \in V$
is a linear operator, known as zero linear operator.

<u>Reflection</u>:- The mapping $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(a_1, a_2, a_3) = (a_1, a_2, -a_3)$, the mapping reflects the vector (a_1, a_2, a_3) through the xy plane which acts like a mirror, is linear transformation, hence linear operator

(i) $f[(a_1, a_2, a_3) + (b_1, b_2, b_3)]$

$= f(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= (a_1 + b_1, a_2 + b_2, -a_3 - b_3)$

$= (a_1, a_2, -a_3) + (b_1, b_2, -b_3)$

$= f(a_1, a_2, a_3) + f(b_1, b_2, b_3)$

hence $f[(a_1, a_2, a_3) + (b_1, b_2, b_3)]$
$= f(a_1, a_2, a_3) + f(b_1, b_2, b_3)$

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(2) $f(c(a_1,a_2,a_3)) = f(ca_1, ca_2, ca_3)$

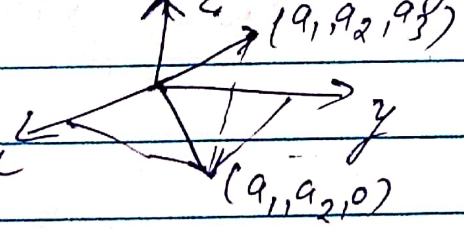
$= (ca_1, ca_2, -ca_3) = c(a_1, a_2, -a_3)$

$= c f(a_1, a_2, a_3)$

hence the above two properties of definition are hold, hence the above reflection is linear transformation here linear operator because domain & codomain are same.

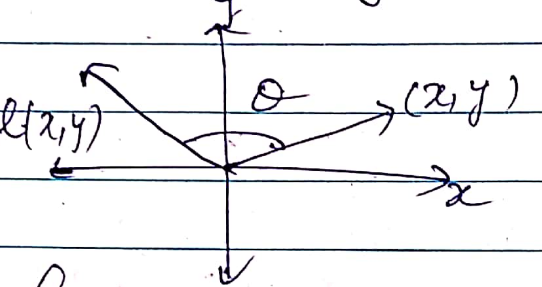
<u>Contraction & Dilations</u>:- The mapping $g: \mathbb{R}^n \to \mathbb{R}^n$ given by $g(x) = kx \ \forall \ x \in \mathbb{R}^n$, $k \in \mathbb{R}$ g is a linear operator. If $|k| > 1$, g represent a dilation of vectors in $\mathbb{R}^n$, if $|k| < 1$, g represent a contraction.

<u>Projection</u>:- The mapping $h: \mathbb{R}^3 \to \mathbb{R}^3$ given by $h(a_1, a_2, a_3) = (a_1, a_2, 0)$. The mapping takes each vector in $\mathbb{R}^3$ to a corresponding vector in the xy-plane



Ex:- The mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by $f(a_1, a_2, a_3, a_4) = (0, a_2, 0, a_4)$. This takes each vector in $\mathbb{R}^4$ to a corresponding vector whose first and third coordinates are zero. The functions f, h are both linear operators. Such mapping where atleast one of the co-ordinates is zeroed out are examples of projection mapping.

Rotation:-



Let θ be a fixed angle in $\mathbb{R}^2$, and let $l : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$l\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\cos\theta + y\sin\theta \end{pmatrix}$$

l rotates (x,y) counterclockwise through the angle θ, and show that it is a linear transformation.

Hence it shows that multiplication by an $m\times n$ matrix is always a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Ex:- Let $A_{m\times n}$, the function $f: \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = Ax$ for $\forall\, x \in \mathbb{R}^n$ is a linear transformation.

Solve:- Let $x_1, x_2 \in \mathbb{R}^n$

(1) $f(x_1+x_2) = A(x_1+x_2)$

$= Ax_1 + Ax_2$

$= f(x_1) + f(x_2)$

(2) $f(cx) = \cancel{cAx} = A(cx) = c(Ax) = cf(x)$

for $A = \begin{pmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{pmatrix}$

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1+4x_2+2x_3 \\ 5x_1+6x_2-3x_3 \end{pmatrix}$$

is a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$. and

The converse of the above result is also true, every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is equivalent to multiplication by an appropriate $m \times n$ matrix.

Theorem: 1 Let V and W be vector spaces, and let $T: V \to W$ be a linear transformation. Let $0 \in V$ be the zero vector in V and 0_W be the zero vector in W. Then

1. ~~$T(0_V) = 0_W$~~ $T(0_V) = 0_W$
2. $T(-v) = -T(v) \ \forall \ v \in V$
3. $T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$
$$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$$
$\forall \ a_1, a_2, \ldots a_n \in \mathbb{R}$ and $v_1, v_2, \ldots v_n \in V$ for $n \geq 2$

Theorem 2 :- Let V_1, V_2, V_3 be vector spaces. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformation. Then $T_2 o T_1 : V_1 \to V_3$ given by
$$(T_2 o T_1)(v) = T_2(T_1(v)), \text{ for all } v \in V_1$$
is a linear transformation.

~~The result is true for~~

The result generalize to more than two linear transformations.

Linear transformation and Subspaces

Theorem:- Let $T: V \to W$ be linear transformation

(1) if V' is a subspace of V, then $T(V') = \{T(v) \mid v \in V'\}$, the image of V' in W is a subspace of W. In particular, the range of T is a subspace of W.

(2) If W' is a subspace of W, then $T^{-1}(W') = \{v \mid T(v) \in W'\}$ the preimage of W' in V, is a subspace of V

Ex:- Let $T: M_{22} \to \mathbb{R}^3$, where $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (b, 0, c)$, T is a linear transformation (Verify). by above theorem range of any linear transformation is a subspace of co-domain. Hence the range of $T = \{(b, 0, c) \mid b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

Also consider the subspace $U_2 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$ of M_{22}. Then the image of U_2 under T is $\{(b,0,0) / b \in \mathbb{R}\}$. this image is a subspace of $\mathbb{R}^3$. finally consider the subspace $W = \{(b, e, 2b) / b, e \in \mathbb{R}\}$ of $\mathbb{R}^3$. The pre-image of W consist of all matrices in M_{22} of the form $\begin{pmatrix} a & b \\ 2b & d \end{pmatrix}$. Note that this preimage is a subspace of M_{22}.

* Students are advised to solve the exercise questions

The matrix of a linear transformation

The behavior of any linear transformation $L: V \to W$ is determined by its effect on a basis for V. In particular when V and W are finite dimensional vector spaces and ordered basis for V and W are choosen, we can obtain a matrix corresponding to L that is useful in computing images under L. Finally; we investigate how the matrix for L changes as the bases for V and W change.

A Linear transformation is determined by its Action on a basis! –

If the action of a linear transformation $L: V \to W$ on a basis for V is known, then the action of L can be computed

~~for all elements of V,~~

Ex!- If $B = \{(0,4,0,1), (-2,5,0,2), (-3,5,1,1), (-1,2,0,1)\}$ is an ordered basis for $\mathbb{R}^4$. Now suppose $L: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation for which

$$L(0,4,0,1) = (3,1,2), \quad L(-2,5,0,2) = (2,-1,1)$$

$$L(-3,5,1,1) = (-4,3,0), \quad L(-1,2,0,1) = (6,1,-1)$$

Then find $L(-4,14,1,4) = ?$

Solve!- Since B is an ordered basis of $\mathbb{R}^4$ hence every vector of $\mathbb{R}^4$ can be written as a linear combination of B, so we can write

$$(-4,14,1,4) = a_1(0,4,0,1) + a_2(-2,5,0,2) + a_3(-3,5,1,1) + a_4(-1,2,0,1)$$

$$\cancel{0a_1 - 2a_2 - 3a_3 - a_4 = -4}$$

$$\cancel{4a_1 + 5a_2 + 5a_3 + 2a_4 = 14}$$

$$0a_1 + 0a_2 + a_3 + 0a_4 = 1$$

$$a_1 + 2a_2 + a_3 + a_4 = 4$$

Solve the system we will get

$$a_1 = 2, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = 3$$

ie $[-4,14,1,4]_B = (2,-1,1,3)$ {by Co-ordinatization}

~~hence~~

$(-4,14,1,4) = 2(0,4,0,1) - 1(-2,5,0,2) + 1(-3,5,1,1) + 3(-1,2,0,1)$

Apply both side L

$L(-4,14,1,4) = L\{2(0,4,0,1) - 1(-2,5,0,2) + 1(-3,5,1,1) + 3(-1,2,0,1)\}$

$= 2L(0,4,0,1) - 1L(-2,5,0,2) + 1L(-3,5,1,1) + 3L(-1,2,0,1)$

Since L is Linear transformation

using the values of $L(0,4,0,1)$, $L(-2,5,0,2)$, $L(-3,5,1,1)$, $L(-1,2,0,1)$ as given in question.

hence

$L(-4,14,1,4) = 2(3,1,2) - (2,-1,1) + (-4,3,0) + 3(6,1,-1)$

$= (18,9,0)$ Ans

hence we can find image of any vector of $\mathbb{R}^4$ under this linear transformation.

In general, if $v \in \mathbb{R}^4$ and $[v]_B = (k_1, k_2, k_3, k_4)$

then $L(v) = k_1(3,1,2) + k_2(2,-1,1) + k_3(-4,3,0) + k_4(6,1,-1)$

$L(v) = (3k_1 + 2k_2 - 4k_3 + 6k_4, k_1 - k_2 + 3k_3 + k_4,$
$2k_1 + k_2 - k_4)$

Thus we derived a general formula for L from its effect on the basis B.

Result:- Let $B = \{v_1, v_2, \ldots v_n\}$ be an ordered basis for a vector space V. Let W be vector space, and let $w_1, w_2, \ldots w_n$ be any n vectors in W. Then there is a unique linear transformation $L: V \to W$ such that $L(v_1) = w_1$, $L(v_2) = w_2$ $\ldots$ $L(v_n) = w_n$.

The matrix of a linear transformation:-

~~Every linear transformation on a finite dimensional vector space can be expressed as a matrix multiplication this will allow us to solve problems involving linear transformations by performing matrix multiplication. The matrix for a linear transformation is determined by the ordered bases B and C chosen for the domain & co-domain respectively. ~~Our~~ Our goal is to find a matrix that ~~to~~ takes the B-co-ordinates of a vector in the domain to the C-co-ordinates of its image vector in the codomain~~

As in the above example the linear transformation $L: \mathbb{R}^4 \to \mathbb{R}^3$ with ordered basis B for $\mathbb{R}^4$. For $v \in \mathbb{R}^4$, we let $[v]_B = [k_1, k_2, k_3, k_4]$ and obtained the formula for L:

$$L(v) = (3k_1 + 2k_2 - 4k_3 + 6k_4,\ k_1 - k_2 + 3k_3 + k_4,\ 2k_1 + k_2 - k_4)$$

Select the standard basis $C = (e_1, e_2, e_3)$ for the co-domain $\mathbb{R}^3$, so that C-co-ordinates of vectors in the co-domain are same as the vectors themselves. This formula for L takes the B-co-ordinates of each vector in the domain to the C-co-ordinates of its image vector in the codomain, Now, notice that if

$$A_{BC} = \begin{pmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix}, \text{ then } A_{BC}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 3k_1 + 2k_2 - 4k_3 + 6k_4 \\ k_1 - k_2 + 3k_3 + k_4 \\ 2k_1 + k_2 - k_4 \end{pmatrix}$$

hence the matrix A contains all of the information needed for carrying out the linear transformation L with respect to the chosen bases B and C.

Result:- Let V and W be ~~nontriv~~ nontrivial vector spaces, with $\dim(V) = n$ and $\dim(W) = m$. Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$ be ordered bases for V and W, respectively. Let $L: V \to W$ be a linear transformation. Then there is a unique $m \times n$ matrix A_{BC} such that $A_{BC}[v]_B = [L(v)]_C \ \forall\, v \in V$.

(That is, A_{BC} times the co-ordinatization of v w.r.t B gives the co-ordinatization of $L(v)$ w.r.t. C)

Furthermore, for $1 \le i \le n$, the ith column for

$$A_{BC} = [L(v_i)]_C$$

~~Hence by above result the matrix A_{BC} is computed~~ as follows: find the image of each domain basis element v_i in turn, and then express these images in C-coordinate to get the respective columns of A_{BC}.

The ~~subscripts B and~~.

__Ex:-__ Find the matrix for L.T, $L: P_3 \to \mathbb{R}^3$ given by

$$L(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_0 + a_1, 2a_2, a_3 - a_0)$$

w.r.t standard basis $B = (x^3, x^2, x, 1)$ for P_3 and

$C = (e_1, e_2, e_3)$ for $\mathbb{R}^3$

Solve:- First we need to find image of each ~~basis~~ vectors of B

$L(x^3) = (0, 0, 1)$, $L(x^2) = (0, 2, 0)$, $L(x) = (1, 0, 0)$

$L(1) = (1, 0, -1)$

using the standard basis C for $\mathbb{R}^3$, each of these images in $\mathbb{R}^3$ is its own C-coordinatization. Then the matrix A_{BC} for L is the matrix whose columns are these images: that is

(16)

$$A_{BC} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

& Compute $L(5x^3 - x^2 + 3x + 2) = ?$

. Since B is the basis of P_3, so we can write linear of B to each vector of P_3

hence

$$5x^3 - x^2 + 3x + 2 = a_1 x^3 + a_2 x^2 + a_3 x + a_4 .$$

$$a_1 = 5,\ a_2 = -1,\ a_3 = 3,\ a_4 = 2$$

$$[5x^3 - x^2 + 5x + 2]_B = (5, -1, 3, 2)$$

$$L(5x^3 - x^2 + 3x + 2) = 5L(x^3) + (-1)L(x^2) + 3L(x) + 2L(1)$$

$$= 5(0,0,1) + (-1)(0,2,0) + 3(1,0,0) + 2(1,0,-1)$$

$$= (5, -2, 3)$$

& Also we can find $[L(5x^3 - x^2 + 3x + 2)]_C$

$$= A_{BC}[v]_B$$

$$[L(5x^3 - x^2 + 3x + 2)]_C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

Ex: Find the matrix for the linear transformation $L: P_3 \to \mathbb{R}^3$ w.r.t. different ordered bases

$$D = \{x^3+x^2, x^2+x, x+1, 1\}$$

$$\& \; E = \{(-2,1,-3), (1,-3,0), (3,-6,2)\}$$

are the bases of P_3 & $\mathbb{R}^4$ respectively.

also find $L(5x^3-x^2+3x+2) = ?$

& $[L(5x^3-x^2+3x+2)]_E = ?$

Finding the new matrix for a linear transformation after a change of Basis:—

Result:- Let V and W be finite dimensional vector spaces with ordered bases B and C respectively. Let $L: V \to W$ be a linear transformation with matrix A_{BC} w.r.t B and C. Suppose that D & E are other ordered bases for V & W respectively. Let P be the transition matrix from B to D and let Q be transition matrix from C to E. Then the matrix A_{DE} for L w.r.t bases D & E is given by $A_{DE} = Q A_{BC} P^{-1}$.

Ex:- The linear transformation

$L: P_3 \to \mathbb{R}^3$ is given by

$L(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_0 + a_1, 2a_2, a_3 - a_0)$

Find the matrix A_{DE} if

$B = \{x^3, x^2, x, 1\}$, $C = \{e_1, e_2, e_3\}$

$D = \{x^3 + x^2, x^2 + x, x + 1, 1\}$ $E = \{(-2, 1, -3), (1, -3, 0), (3, -6, 2)\}$

are the bases of P_3 and $\mathbb{R}^3$ respectively.

Solve:- From the previous example, the matrix

$$A_{BC} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

the transition matrix Q from bases C to E is

$$Q = \begin{pmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{pmatrix} \quad (\text{verify})$$

Also the transition matrix P^{-1} from D to B is

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (\text{verify})$$

Hence

$$A_{DE} = Q A_{BC} P^{-1} = \begin{pmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A_{DE} = \begin{pmatrix} -1 & -10 & -15 & -19 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{pmatrix}$$

Linear operators and Similarity

Suppose L is a linear operator on a finite dimensional vector space V. If B is a basis for V. Then there is some matrix A_{BB} for L w.r.t B. Also, if C is another basis for V, then there is some matrix A_{CC} for L w.r.t C. Let P be the transition matrix from B to C, hence by previous result we have $A_{BB} = P^{-1} A_{CC} P$ and so by definition of similar matrices, A_{BB} & A_{CC} are similar. Hence any two matrices for the same linear operator w.r.t different bases are similar.

Ques.- The linear operator $L: \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix w.r.t standard basis B for $\mathbb{R}^3$ is

$$A_{BB} = \frac{1}{7}\begin{pmatrix} 6 & 3 & -2 \\ 3 & -2 & 6 \\ -2 & 6 & 3 \end{pmatrix}$$

If $D = \{(3,1,0), (-2,0,1), (1,-3,2)\}$ be an another ordered basis for $\mathbb{R}^3$ then find $A_{DD} = ?$

Solve:- Since A_{BB} & A_{DD} are similar, so $\exists$ a nonsingular matrix P s.t

$$A_{DD} = P^{-1} A_{BB} P$$

Where P is the transition matrix from D to B

$$P = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{(verify)}$$

$$P^{-1} = \frac{1}{14}\begin{pmatrix} 3 & 5 & 6 \\ -2 & 6 & 10 \\ 1 & -3 & 2 \end{pmatrix} \quad \text{(verify)}$$

$$A_{DD} = P^{-1} A_{BB} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which is required result

Ans-

Matrix for the composition of Linear operators:-

Result:- Let V_1, V_2 and V_3 be non trivial finite dimensional vector spaces, with ordered basis B, C and D respectively, let $L_1 : V_1 \to V_2$ be L.T with A_{BC} w.r.t B and C, and let $L_2 : V_2 \to V_3$ be a L.T with matrix A_{CD} w.r.t C and D. then the matrix A_{BD} for the composite L.T. $L_2 \circ L_1 : V_1 \to V_3$ w.r.t bases B and D is the product $A_{CD} A_{BC}$.

Kernel, Range and dimension theorem

Kernel of a linear transformation:-

Definition:- Let $L: V \to W$ be a L.T. The Kernel of L, denoted by $\ker(L)$, is the subset of all vectors in V that map to 0_W. That is $\ker(L) = \{ v \in V \mid L(v) = 0_W \}$. The range of L, or range(L), is the subset of all vectors in W that are the image of some vector in V. That is

$$\text{range}(L) = \{ L(v) \mid v \in V \}$$

Kernel is a subset of the domain and that the range is a subset of codomain. Since the Kernel of $L: V \to W$ is the pre-image of the subspace $\{0_W\}$ of W, it must be subspace of V and range of L is subspace of W.

Result:- If $L: V \to W$ be a L.T, then Kernel of L is a subspace of V and the range of L is a subspace of W.

Ex:- Consider the linear operator $L: \mathbb{R}^n \to \mathbb{R}^n$ given by $L(a_1, a_2, \ldots a_n) = (a_1, a_2, 0, \ldots 0)$ find $\ker(L) = ?$

Soln:- Ker(L) = { consists of those elements of the domain that map to {(0, 0, - - 0)} the zero vector of the domain. hence for vectors in the kernel $a_1 = 0, a_2 = 0$ but $a_3, a_4 - - a_n$ can have any values thus

$$\ker(L) = \{ (0, 0, a_3 - - a_n) / a_3, - - a_n \in \mathbb{R} \}$$

$$\dim(\ker(L)) = n-2$$

because the standard basis vectors $e_3, - - e_n$ of $\mathbb{R}^n$ span Ker(L)

Range(L) consists of those elements of the codomain $\mathbb{R}^n$ that the images of domain elements, Hence

$$\text{Rang}(L) = \{ (a_1, a_2, 0, - - 0) / a_1, a_2 \in \mathbb{R} \}$$

$$\dim(\text{Rang}(L)) = 2$$

standard basis e_1 & e_2 span Range

Ex:- The linear transformation $L : P_3 \to P_2$ given by $L(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$

find Ker(L) & range(L)

Solve:- $Ker(L) = \{ 3ax^2 + 2bx + c = 0$
we must have $a = b = c = 0$

$Ker(L) = \{ 0x^3 + 0x^2 + 0x + d \mid d \in \mathbb{R} \}$

$dim(Ker(L)) = 1$

$Range(L) = ?$

Finding the kernel from the matrix of a linear transformation

Consider the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(X) = AX$ where $A_{m \times n}$ $X \in \mathbb{R}^n$, the $Ker(L)$ is a subspace of all vectors X in the domain that are solution of homogeneous system $AX = 0$

Method of finding a basis for kernel of a L.T. (Kernel method):-

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(X) = AX$ for some $m \times n$ matrix A. To find a basis for $Ker(L)$, perform the following steps:

~~Step ①:- find B, reduced row echelon form of A~~

Step ②:- Solve for one particular solution for each independent variable in the homogeneous system $BX = 0$. The ith such solution v_i, is found by setting the ith independent variable equal to 1 and setting all other independent variables equal to zero.

Step ③:- The set $\{v_1, v_2, \ldots v_k\}$ is a basis for $\mathrm{Ker}(L)$.
(We can replace any v_i with cv_i, where $c \neq 0$ to eliminate fractions.)

Finding a Basis for the Range of a L.T (Range Method)

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $L(X) = AX$ for some $m \times n$ matrix A. To find a basis for range (L). Perform the following steps:

Step ①:- Find B, the reduced row echelon form of A

Step ②:- Form the set of those columns of A whose corresponding columns in B have nonzero pivots. ~~Thus~~ This set is a basis for range (L).

Que:- Let $L : \mathbb{R}^5 \to \mathbb{R}^4$ given by $L(X) = AX$, where

$$A = \begin{pmatrix} 8 & 4 & 16 & 32 & 0 \\ 4 & 2 & 10 & 22 & -4 \\ -2 & -1 & -5 & -11 & 7 \\ 6 & 3 & 15 & 33 & -7 \end{pmatrix}$$

find Ker(L) & rang(L)

Solve:- reduced row echelon form B of A is

$$B = \begin{pmatrix} 1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Second & four column having no pivot hence x_2 & x_4 are free variables

for $x_2 = 1,\ x_4 = 0,$

for $x_2 = 0,\ x_4 = 1$

$v_1 = (-\frac{1}{2}, 1, 0, 0, 0)$, $v_2 = (2, 0, -3, 1, 0)$ are two independent vectors and hence basis of Ker(L)

(26)

~~hence~~ $\dim(\ker(L)) = 2$

$\ker(L) = \{ a_1 x_1 + a_2 x_2 \mid a_1, a_2 \in \mathbb{R} \}$

$= \{ (-4a_1 + 2a_2, a_1, -3a_2, 6, 0) \mid a_1, a_2 \in \mathbb{R} \}$

$Range(L) = \{ (8,4,-2,6), (16,10,-5,15), (0,-4,7,-7) \}$

because 1^{st}, 3rd & 5^{th} column having pivot.

hence the 1^{st}, 3^{rd} & 5^{th} columns of A are ~~range elements~~ basis of Range by Range method.

and $\dim(Range(L)) = \underline{3}$

$Range(L) = \{ a(8,4,-2,6) + b(16,10,-5,15) + c(0,-4,7,-7) \mid a, b, c \in \mathbb{R} \}$

Ans

Theorem:- If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T with matrix A w.r.t. any bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, then

① $\dim(\text{Range}(L)) = \text{rank}(A)$

② $\dim(\text{Ker}(L)) = n - \text{rank}(A)$

③ $\dim(\text{Ker}(L)) + \dim(\text{Rang}(L))$

$= \dim(\text{domain}(L)) = n$

Dimension theorem:- If $L: V \to W$ be a L.T and V is finite dimensional, then range(L) is finite dimensional, and

$$\boxed{\dim(\text{Ker}(L)) + \dim(\text{Rang}(L)) = \dim(V)}$$

Result:- If A is any matrix, then

$$\text{rank}(A) = \text{rank}(A^T)$$

Ex:-

$$A = \begin{pmatrix} 8 & 4 & 16 & 32 & 0 \\ 4 & 2 & 10 & 22 & -4 \\ -2 & -1 & -5 & -11 & 7 \\ 6 & 3 & 15 & 33 & -7 \end{pmatrix}$$

$\text{rank}(A) = 3$ (verify)

$$A^T = \begin{pmatrix} 8 & 4 & -2 & 6 \\ 4 & 2 & -1 & 3 \\ 16 & 10 & -5 & 15 \\ 32 & 22 & -11 & 33 \\ 0 & -4 & 7 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A^T) = 3$

hence $\boxed{\text{rank}(A) = \text{rank}(A^T)}$

(*) Students are Advised to Solve exercise Questions'

One-To One and Onto Linear transformation

Definition:- Let $L: V \to W$ be a Linear transformation

1. L is one-to-one iff distinct vectors in V have different images in W. That is L is one-to-one iff $\forall\ v_1, v_2 \in V$, $L(v_1) = L(v_2)$ $\Rightarrow v_1 = v_2$

2. L is onto iff every vector in the codomain W is the image of some vector in the domain V. That is L is onto iff, for every $w \in W$, there is some $v \in V$ s.t $L(v) = w$

Ex:- $L: P_3 \to P_2$ given by

$L(P) = P'$ is onto but ~~one-one~~

For one-one:- Take $P_1 = x+1$, $P_2 = x+2$

$L(P_1) = L(P_2) = 1$

hence L is not one-one

To L is onto we must take a vector q in P_2 and find some vector p in P_3 s.t $L(P) = q$

Consider the vector $P = \int q(x)dx$ with zero constant. because $L(P) = q$, we see that L is onto.

Theorem:- Let V and W be vector spaces, and Let $L : V \to W$ be a L.T. Then

① L is one-one iff $Ker(L) = 0$ (or equivalently, iff $\dim(Ker(L)) = 0$), and

② if W is finite dimensional, then L is onto iff $\dim(Range(L)) = \dim(W)$

Ex!- Consider the linear transformation $L : M_{22} \to M_{23}$ given by $L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{pmatrix}$.

Show that it is one-one but not onto

Soln!- A linear transformation is one-one iff its $Ker(L) = 0$

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Ker(L)$

Then $a-b = c-d = c+d = a+b = 0$

Solving these eq^ns $a=b=c=d=0$

So $Ker(L)$ contains only zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

~~hence $\dim(\ker(L)) = 0$~~

hence one-one.

by dimension theorem

$$\dim(\text{Range}(L)) = \dim(M_{22}) - \dim(\ker L)$$
$$= 4 - 0 = 4$$

For onto, in particular

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin \text{Range}(L)$$

hence not onto.

~~On the other hand consider $M : M_{23} \to M_{22}$~~
given by $M\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{pmatrix} a+b & a+c \\ d+e & d+f \end{pmatrix}$

Since $M\left(\begin{bmatrix} 0 & b & c \\ 0 & e & f \end{bmatrix}\right) = \begin{pmatrix} b & c \\ e & f \end{pmatrix}$

~~and thus every~~ 2×2 matrix is in range(M)

$$\dim(\text{range}\, M) = \dim(M_{22}) = 4$$

$\dim(\ker M) = 6 - 4 = 2$
M is not one-one

in particular $\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \in \ker(L)$

Theorem:- ~~Let V and W be vector spaces, and let $L: V \to W$ be a linear transformation. Then~~

① If L is one-one, and T is a linearly ~~independ~~ independent subset of V, then $L(T)$ is Linearly independent in W

② If L is onto, and S ~~spans V~~ spans V, then $L(S)$ spans W.

Example:- Consider the linear transformation $L: P_2 \to P_3$ given by

$L(ax^2+bx+c) = bx^3+cx^2+ax$

it is see that $Ker(L)=0$

Since $L(ax^2+bx+c) = 0x^3+0x^2+0x+0$

only if $a=b=c=0$

and so L is one-one

Consider the L.I set $T=\{x^2+x, x+1\}$ in P_2. Note that $L(T)=\{x^3+x, x^3+x^2\}$

and that $L(T)$ is L.I

~~let $W=\{(x,0,z)\}$ be the xz-plane in $\mathbb{R}^3$~~

$\dim(W)=2$, consider $L: \mathbb{R}^3 \to W$

Where L is the projection of $\mathbb{R}^3$ onto xz plane

i.e $L(x,y,z) = (x,0,z)$

$S = \{(2,-1,3),(1,-2,0),(4,3,-1)\}$ spans $\mathbb{R}^3$

$L(S) = \{(2,0,3),(1,0,0),(4,0,-1)\}$ span W

In fact $\{(2,0,3),(1,0,0)\}$ ~~also~~ alone spans W, since span $((2,0,3),(1,0,0)) = 2 = \dim(W)$

Isomorphism

↓

Invertible linear transformation

└ one-one & onto linear transformation

Definition:- Let $L: V \to W$ be a linear transformation. Then L is an invertible linear transformation iff there is a function $M: W \to V$ such that
$(M \circ L)(v) = v \;\forall\; v \in V$
and $(L \circ M)(w) = w \;\forall\; w \in W$. Such a function M is called an inverse of L.

If the inverse M of $L: V \to W$ exists. Then it is unique

Definition:- A linear transformation $L: V \to W$ that is both one-one & onto is called isomorphism from V to W.

Theorem:- Let $L: V \to W$ be a linear transformation, then L is an isomorphism iff L is an invertible linear transformation. Moreover, if L is invertible then L^{-1} is also a linear transformation.

Example:- The linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ with

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is an invertible linear operator. (Verify)

hence isomorphism from $\mathbb{R}^2 \to \mathbb{R}^2$

or L one-one & onto linear operator

Theorem:- Let V and W be nontrivial finite dimensional vector spaces with ordered bases B and C, respectively and let $L: V \to W$ be a linear transformation. Then L is an ~~isom~~ isomorphism iff the matrix representation A_{BC} for L w.r.t B and C is nonsingular.

Example:- Consider $L : \mathbb{R}^3 \to \mathbb{R}^3$
given by $L(v) = Av$
where $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

Now, A is nonsingular (ie $|A| \neq 0$)
hence by above theorem
L is an isomorphism.

Theorem:- Suppose $L : V \to W$ is an isomorphism. Let S span V and let T be a linearly independent subset of V. Then $L(S)$ spans W and $L(T)$ is linearly independent.

Isomorphic Vector spaces:-

Let V and W be vector spaces. Then V is isomorphic to W, denoted by $V \cong W$ iff $\exists$ an isomorphism $L : V \to W$

Example:- Consider $L_1 : \mathbb{R}^4 \to P_3$
given $L_1(a,b,c,d) = ax^3 + bx^2 + cx + d$
& $L_2 : M_{22} \to P_3$ given by
$L_2\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = ax^3 + bx^2 + cx + d$

here L_1 & L_2 both are one-one and onto (Verify)
hence both are invertible, that is isomorphisms.

~~hence by above theorem~~

$$\mathbb{R}^4 \cong P_3 \quad \& \quad M_{22} \cong P_3$$

Thus the composition $L_2^{-1} \circ L_1 : \mathbb{R}^4 \to M_{22}$ is also an isomorphism, and so

$$\mathbb{R}^4 \cong M_{22}$$

Notice that all vector spaces have dimension 4

Theorem: (1) Suppose $V \cong W$ and V is finite dimensional. Then W is finite dimensional and $\dim(V) = \dim(W)$.

(2) If V is any n-dimensional vector space, then

$$V \cong \mathbb{R}^n$$

(3) Any two n-dimensional vector spaces V and W are isomorphic. That is if $\dim(V) = \dim(W)$ then $V \cong W$

(4) Let V and W be finite dimensional vector spaces with $\dim(V) = \dim(W)$. Let $L : V \to W$ be a linear transformation. Then L is one-one iff L is onto.

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