SUMMARY

Linear differential equation with constant coefficients

Form: $$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)}$$

(a_i's are constants)

The solution is given by $y = C.F. + P.I.$

C.F. is the solution of the associated homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Method for finding C.F.

Find the Auxiliary Equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0.$$

Let $r_1, r_2, \dots r_n$ be the roots.

1) If $r_1, r_2, \dots r_n$ are distinct & real then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

is the general solution.

2) If r_1 is a root repeated k times then the part of a general solution corresponding to r_1 is of the form

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{r_1 x}$$

3) If the characteristic eqⁿ / Auxiliary eqⁿ has an unrepeated pair of complex conjugate roots $a \pm bi$ $(b \neq 0)$ then the corresponding part of a general solution has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

If the pair is repeated twice then

$$e^{ax}\left((c_1+c_2x)\cos bx + (c_3+c_4x)\sin bx\right)$$

If the pair is repeated thrice then

$$e^{ax}\left((c_1+c_2x+c_3x^2)\cos bx + (c_4+c_5x+c_6x)\sin bx\right)$$

and so on.

Examples on finding C.F.

1) $y'' + 4y = 0$.

A.E : $r^2 + 4 = 0$

$r = 0 \pm 2i$ (complex pair, unrepeated)

$\therefore y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x$.

2). $y'' - 4y' = 0$

A.E $r^2 - 4r = 0 \Rightarrow r = 0, 4$ (real, distinct roots)

$\therefore \quad y = c_1 e^{0x} + c_2 e^{4x}$

$\Rightarrow \quad y = c_1 + c_2 e^{4x}$

3) Solve the IVP $\quad y'' - 4y' + 3y = 0$
$y(0) = 7$
$y'(0) = 11$

A.E: $\quad r^2 - 4r + 3 = 0$

$r^2 - 3r - r + 3 = 0$

$r = 3, 1$

$\therefore \quad y = c_1 e^{x} + c_2 e^{3x}$

Given $\quad y(0) = 7 \Rightarrow c_1 + c_2 = 7$

$y'(0) = 11 \Rightarrow \boxed{c_1 + 3c_2 = 11}$

$\Rightarrow c_1 = 5, \; c_2 = 7$

$\therefore \quad y = 5e^{x} + 2e^{3x}$ is the solution

4). $\quad 9y^{(3)} + 12y'' + 4y' = 0$

A.E. $\quad 9r^3 + 12r^2 + 4r = 0$

$\Rightarrow \quad r = 0, \quad 9r^2 + 12r + 4 = 0$

$\Rightarrow \quad \underbrace{r = 0}_{\text{distinct real}}, \underbrace{-\frac{2}{3}, -\frac{2}{3}}_{\text{real repeated twice}}$

$\therefore \quad y = c_1 e^{0x} + (c_2 + c_3 x) e^{-2/3 x} \Rightarrow y = c_1 + (c_2 + c_3 x) e^{-2/3 x}$

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(5) $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$

A.E. $r^4 + 2r^3 + 3r^2 + 2r + 1 = 0$

$\Rightarrow (r^2 + r + 1)^2 = 0$

$\Rightarrow r = \left(\frac{-1 \pm \sqrt{1-4}}{2}\right), \left(\frac{-1 \pm \sqrt{1-4}}{2}\right)$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

(complex pair repeated twice)

$\therefore$ Sol^n is

$$y = e^{-\frac{1}{2}x}\left((c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right)$$

Assignment.

Q1 – 42 of Ex 2.3.

5\—30

Find the general solutions of the differential equations in Problems 1 through 20.

1. $y'' - 4y = 0$
2. $2y'' - 3y' = 0$
3. $y'' + 3y' - 10y = 0$
4. $2y'' - 7y' + 3y = 0$
5. $y'' + 6y' + 9y = 0$
6. $y'' + 5y' + 5y = 0$
7. $4y'' - 12y' + 9y = 0$
8. $y'' - 6y' + 13y = 0$
9. $y'' + 8y' + 25y = 0$
10. $5y^{(4)} + 3y^{(3)} = 0$
11. $y^{(4)} - 8y^{(3)} + 16y'' = 0$
12. $y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$
13. $9y^{(3)} + 12y'' + 4y' = 0$
14. $y^{(4)} + 3y'' - 4y = 0$
15. $y^{(4)} - 8y'' + 16y = 0$
16. $y^{(4)} + 18y'' + 81y = 0$
17. $6y^{(4)} + 11y'' + 4y = 0$
18. $y^{(4)} = 16y$
19. $y^{(3)} + y'' - y' - y = 0$
20. $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$ (*Suggestion:* Expand $(r^2 + r + 1)^2$.)

Solve the initial value problems given in Problems 21 through 26.

21. $y'' - 4y' + 3y = 0$; $y(0) = 7$, $y'(0) = 11$
22. $9y'' + 6y' + 4y = 0$; $y(0) = 3$, $y'(0) = 4$
23. $y'' - 6y' + 25y = 0$; $y(0) = 3$, $y'(0) = 1$
24. $2y^{(3)} - 3y'' - 2y' = 0$; $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$
25. $3y^{(3)} + 2y'' = 0$; $y(0) = -1$, $y'(0) = 0$, $y''(0) = 1$
26. $y^{(3)} + 10y'' + 25y' = 0$; $y(0) = 3$, $y'(0) = 4$, $y''(0) = 5$

Find general solutions of the equations in Problems 27 through 32. First find a small integral root of the characteristic equation by inspection; then factor by division.

27. $y^{(3)} + 3y'' - 4y = 0$
28. $2y^{(3)} - y'' - 5y' - 2y = 0$
29. $y^{(3)} + 27y = 0$
30. $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$
31. $y^{(3)} + 3y'' + 4y' - 8y = 0$
32. $y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

In Problems 33 through 36, one solution of the differential equation is given. Find the general solution.

33. $y^{(3)} + 3y'' - 54y = 0$; $y = e^{3x}$
34. $3y^{(3)} - 2y'' + 12y' - 8y = 0$; $y = e^{2x/3}$
35. $6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0$; $y = \cos 2x$
36. $9y^{(3)} + 11y'' + 4y' - 14y = 0$; $y = e^{-x} \sin x$

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The roots of the characteristic equation of a certain differential equation are 3, −5, 0, 0, 0, 0, −5, $2 \pm 3i$, and $2 \pm 3i$. Write a general solution of this homogeneous differential equation.

The solution can be read directly from the list of roots. It is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{3x} + c_6e^{-5x} + c_7xe^{-5x} + e^{2x}(c_8\cos 3x + c_9\sin 3x) + xe^{2x}(c_{10}\cos 3x + c_{11}\sin 3x).$$ ■

Q1-42
51-58

...ferential equations in Prob-

2. $2y'' - 3y' = 0$
4. $2y'' - 7y' + 3y = 0$
6. $y'' + 5y' + 5y = 0$
8. $y'' - 6y' + 13y = 0$
0. $5y^{(4)} + 3y^{(3)} = 0$

4. $y^{(4)} + 3y'' - 4y = 0$
16. $y^{(4)} + 18y'' + 81y = 0$
18. $y^{(4)} = 16y$

$y = 0$ (*Suggestion:* Expand

...iven in Problems 21 through

37. Find a function $y(x)$ such that $y^{(4)}(x) = y^{(3)}(x)$ for all x and $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, and $y^{(3)}(0) = 7$.

38. Solve the initial value problem

$$y^{(3)} - 5y'' + 100y' - 500y = 0;$$
$$y(0) = 0, \quad y'(0) = 10, \quad y''(0) = 250$$

given that $y_1(x) = e^{5x}$ is one particular solution of the differential equation.

In Problems 39 through 42, find a linear homogeneous constant-coefficient equation with the given general solution.

39. $y(x) = (A + Bx + Cx^2)e^{2x}$
40. $y(x) = Ae^{2x} + B\cos 2x + C\sin 2x$
41. $y(x) = A\cos 2x + B\sin 2x + C\cosh 2x + D\sinh 2x$
42. $y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$

... solution of differential