Theorem 2: Fundamental Theorem
(Existence & Uniqueness for linear equations)
[Statement only].

Suppose that the functions p, q and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)\, y' + q(x)\, y = f(x) \quad \text{——} \quad (1)$$

has a unique solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1 .$$

Remark 1: The above theorem states that a second order linear IVP having continuous coefficients has a unique solution on the whole interval I.

Remark 2: Suppose that $y(x)$ is a solution of $y'' + p(x)y' + q(x)y = 0$ that satisfies $y(a) = 0$, $y'(a) = 0$ then $y(x) \equiv 0$ on I.

Proof: We have $Y(x) \equiv 0$ as a solution of $y'' + p(x)\, y' + q(x)\, y = 0$ that satisfies $Y(a) = 0$ & $Y'(a) = 0$. So by uniqueness part of above theorem we have $y(x) = Y(x)$ on I

$$\Rightarrow y(x) \equiv 0 \text{ on } I$$

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Wronskian of two functions f and g is defined as $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$

trivial i.e. f

Example $W(e^x, \cos x) = \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix}$

$= -e^x \sin x - e^x \cos x$

$= -e^x(\sin x + \cos x)$.

Theorem 3: Wronskian of solutions

Suppose y_1 & y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

on I where p, q are continuous functions on I.

(a) If y_1 & y_2 are L.D. on I then $W(y_1, y_2) \equiv 0$ on I.

(b) If y_1 & y_2 are L.I. then $W(y_1, y_2) \neq 0$ at each pt. of I.

Proof: (a) Let y_1 and y_2 be L.D. on I.

Then $y_1 = k y_2$ on I, for some real no. k.

$\therefore$ $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} ky_2 & y_2 \\ ky_2' & y_2' \end{vmatrix} = 0$

$\Rightarrow W(y_1, y_2) \equiv 0$ on I.

(b) Assume that $W(y_1, y_2) = 0$ at some point $a \in I$.

We show that y_1 & y_2 are L.D. on I.

Consider the system of equations: $\left.\begin{array}{l} c_1 y_1(a) + c_2 y_2(a) = 0 \\ c_1 y_1'(a) + c_2 y_2'(a) = 0 \end{array}\right\}$

in unknowns c_1 & c_2.

Since $W(a) = 0$, by Linear Algebra; $\exists$ a non-trivial solution of above system in c_1 & c_2.

i.e $\exists$ c_1, c_2 (atleast one non-zero) such that

$c_1 y_1(a) + c_2 y_2(a) = 0$. Using these c_1 & c_2 we define

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \text{on } I.$$

Clearly $Y(x)$ is a solution of $y'' + p(x) y' + q(x) y = 0$ (by principle of superposition). Also $Y(x)$ satisfies

$$Y(a) = 0 = Y'(a)$$

$\therefore$ $Y(x) \equiv 0$ on I (by Remark 2)

$\Rightarrow$ $c_1 y_1(x) + c_2 y_2(x) = 0$ on I, where c_1 & c_2 are not both zero.

$\Rightarrow$ y_1 & y_2 are L.D. on I.

<u>Theorem 4</u> : <u>General solution of Homogeneous equations</u>:

Let y_1 & y_2 be two linearly independent solutions of the homogeneous equation $y'' + p(x) y' + q(x) y = 0$; with p and q continuous on the open interval I.

If Y is any solution whatsoever of this eq^n on I, then $\exists$ numbers c_1 & c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall\ x \in I.$$

<u>Proof</u>: Let $Y(x)$ be any solution of $y'' + p(x) y' + q(x) y = 0$.

Choose $a \in I$. Consider the system:

$$\left.\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= Y(a) \\ c_1 y_1'(a) + c_2 y_2'(a) &= Y'(a) \end{aligned}\right\}$$ for unknowns c_1 & c_2.

The determinant of coeffs. of this system is

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = W(y_1, y_2) \text{ at } a$$

$$\neq 0 \qquad (\because y_1 \text{ \& } y_2 \text{ are L.I. on } I)$$

So by elementary algebra it follows that the system has a solution in c_1 & c_2. With these values of c_1 & c_2 we define

$$G(x) = c_1 y_1(x) + c_2 y_2(x) \text{ on } I.$$

Then
$$G(a) = c_1 y_1(a) + c_2 y_2(a) = Y(a)$$
$$G'(a) = c_1 y_1'(a) + c_2 y_2'(a) = Y'(a)$$

$$\Rightarrow \quad Y(x) \equiv G(x) \text{ on } I$$

$$\Rightarrow \quad Y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ on } I.$$

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