DEFINITION Linear Independence of Two Functions

Two functions defined on an open interval I are said to be **linearly independent** on I provided that neither is a constant multiple of the other.

Two functions are said to be **linearly dependent** on an open interval provided that they are not linearly independent there; that is, one of them is a constant multiple of the other. We can always determine whether two given functions f and g are linearly dependent on an interval I by noting at a glance whether either of the two quotients f/g or g/f is a constant-valued function on I.

Example 3 Thus it is clear that the following pairs of functions are linearly independent on the entire real line:

$$\begin{aligned} \sin x \quad &\text{and} \quad \cos x; \\ e^x \quad &\text{and} \quad e^{-2x}; \\ e^x \quad &\text{and} \quad xe^x; \\ x+1 \quad &\text{and} \quad x^2; \\ x \quad &\text{and} \quad |x|. \end{aligned}$$

That is, neither $\sin x/\cos x = \tan x$ nor $\cos x/\sin x = \cot x$ is a constant-valued function; neither $e^x/e^{-2x} = e^{3x}$ nor e^{-2x}/e^x is a constant-valued function; and so forth. But the identically zero function $f(x) \equiv 0$ and any other function g are linearly dependent on every interval, because $0 \cdot g(x) = 0 = f(x)$. Also, the functions

$$f(x) = \sin 2x \quad \text{and} \quad g(x) = \sin x \cos x$$

are linearly dependent on any interval because $f(x) = 2g(x)$ is the familiar trigonometric identity $\sin 2x = 2 \sin x \cos x$. ■

Determine whether the pairs of functions in Problems 20 through 26 are linearly independent or linearly dependent on the real line.

20. $f(x) = \pi$, $g(x) = \cos^2 x + \sin^2 x$ LD
21. $f(x) = x^3$, $g(x) = x^2|x|$ LI
22. $f(x) = 1 + x$, $g(x) = 1 + |x|$ LI
23. $f(x) = xe^x$, $g(x) = |x|e^x$ LI
24. $f(x) = \sin^2 x$, $g(x) = 1 - \cos 2x$ LD
25. $f(x) = e^x \sin x$, $g(x) = e^x \cos x$ LI
26. $f(x) = 2\cos x + 3\sin x$, $g(x) = 3\cos x - 2\sin x$ LI