→ (Statement only)

* Theorem 2 : Fundamental Theorem of Existence and Uniqueness for linear equations

Suppose that the functions p, q and f are continuous on the open interval I containing the point 'a'. Then given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{— (A)}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \quad \text{— (B)}$$

Remark: Equation (A) and conditions (B) put together constitute a second order linear Initial Value Problem. Theorem 2 tells us that any such ~~which~~ IVP has a unique solution.

Example: Verify that the functions $y_1(x) = e^x$, $y_2(x) = xe^x$ are solutions of the differential equation

$$y'' - 2y' + y = 0$$

and then find a solution satisfying the initial conditions $y(0) = 3$, $y'(0) = 1$.

Solution: (1) Verify that $y_1(x) = e^x$, $y_2(x) = xe^x$ are solutions. (DIY)

(2) We impose the given conditions on the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

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ie $y(x) = c_1 e^x + c_2 x e^x$

Given $y(0) = 3 \Rightarrow c_1 = 3$

$y'(0) = 1 \Rightarrow c_1 + c_2 = 1$

$\Rightarrow c_1 = 3, \; c_2 = -2$

$\therefore$ Reqd. solution is $y(x) = 3e^x - 2xe^x$

Assignment

Q1 — 16 of Exercise 2.1.

2.1 Problems

81–48
51–56

In Problems 1 through 16, a homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

1. $y'' - y = 0$; $y_1 = e^x$, $y_2 = e^{-x}$; $y(0) = 0$, $y'(0) = 5$
2. $y'' - 9y = 0$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$; $y(0) = -1$, $y'(0) = 15$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$; $y(0) = 3$, $y'(0) = 8$
4. $y'' + 25y = 0$; $y_1 = \cos 5x$, $y_2 = \sin 5x$; $y(0) = 10$, $y'(0) = -10$
5. $y'' - 3y' + 2y = 0$; $y_1 = e^x$, $y_2 = e^{2x}$; $y(0) = 1$, $y'(0) = 0$
6. $y'' + y' - 6y = 0$; $y_1 = e^{2x}$, $y_2 = e^{-3x}$; $y(0) = 7$, $y'(0) = -1$

7. $y'' + y' = 0$; $y_1 = 1$, $y_2 = e^{-x}$; $y(0) = -2$, $y'(0) = 8$
8. $y'' - 3y' = 0$; $y_1 = 1$, $y_2 = e^{3x}$; $y(0) = 4$, $y'(0) = -2$
9. $y'' + 2y' + y = 0$; $y_1 = e^{-x}$, $y_2 = xe^{-x}$; $y(0) = 2$, $y'(0) = -1$
10. $y'' - 10y' + 25y = 0$; $y_1 = e^{5x}$, $y_2 = xe^{5x}$; $y(0) = 3$, $y'(0) = 13$
11. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$; $y(0) = 0$, $y'(0) = 5$
12. $y'' + 6y' + 13y = 0$; $y_1 = e^{-3x} \cos 2x$, $y_2 = e^{-3x} \sin 2x$; $y(0) = 2$, $y'(0) = 0$
13. $x^2y'' - 2xy' + 2y = 0$; $y_1 = x$, $y_2 = x^2$; $y(1) = 3$, $y'(1) = 1$
14. $x^2y'' + 2xy' - 6y = 0$; $y_1 = x^2$, $y_2 = x^{-3}$; $y(2) = 10$, $y'(2) = 15$
15. $x^2y'' - xy' + y = 0$; $y_1 = x$, $y_2 = x \ln x$; $y(1) = 7$, $y'(1) = 2$
16. $x^2y'' + xy' + y = 0$; $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$; $y(1) = 2$, $y'(1) = 3$

The following three problems illustrate the fact that the superposition principle does not generally hold for nonlinear equations.

17. Show that $y = 1/x$ is a solution of $y' + y^2 = 0$, but that if $c \neq 0$ and $c \neq 1$, then $y = c/x$ is not a solution.
18. Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.
19. Show that $y_1 \equiv 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.