2 Linear Equations of Higher Order

UNIT - 3

SECTION 1 : Linear equations of higher order

2.1 Second-order linear equations

A general second order linear differential equation is an equation of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x) \quad \text{—} \quad (1)$$

Examples 1) $y'' + \cos x \, y' = 0$

2) $e^x y'' + \cos x \, y' + (1+\sqrt{x})y = \tan^{-1} x$

are linear second order diff. eq^{n}s.

$y'' = yy'$, $y'' + 3(y')^2 + 4y^3 = \cos x$,

$y'' = y^2 y'$ are non-linear differential equations because these equations involve products/ powers of y and its derivatives

So linearity means linearity w.r.t y and its derivatives.

If the function $F(x)$ appearing on RHS is zero, we say that the equation is homogeneous.

Thus a second-order homogeneous differential equation looks like

$$\boxed{A(x)y'' + B(x)y' + C(x)y = 0}$$

Remark: The meaning of the term "homogeneous" for a second order linear diff. eqn is different from its meaning for a first order differential equation

CONSIDER A GENERAL SECOND ORDER LINEAR DIFFERENTIAL EQUATION

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

where the coefficient functions A, B, C and F are continuous on the open interval I. Assume in addition that $A(x) \neq 0$ at each point of I so that we can rewrite the above equation in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where p, q, f are continuous functions on I

We first discuss a result associated to the homogeneous differential equation of the form

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— (A)}$$

where p, q are continuous functions on I

* **Theorem 1: PRINCIPLE OF SUPERPOSITION FOR HOMOGENEOUS EQUATIONS**

Let y_1 and y_2 be two solutions of the homogeneous linear equation (A) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1y_1 + c_2y_2$$

is also a solution of (A) on I.

Pf: As y_1 and y_2 are solutions of (A) we have,

$$\left.\begin{aligned} & y_1'' + p(x)y_1' + q(x)y_1 = 0 \\ \& \quad & y_2'' + p(x)y_2' + q(x)y_2 = 0 \end{aligned}\right] \text{— (*)}$$

We show that $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution of (A) on I.

Consider $y'' + p(x)y' + q(x)y$

$$= \left(c_1y_1(x) + c_2y_2(x)\right)'' + p(x)\left(c_1y_1(x) + c_2y_2(x)\right)' + q(x)\left(c_1y_1(x) + c_2y_2(x)\right)$$

$$= c_1\left(y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\right) + c_2\left(y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)\right)$$

(using algebra of derivatives)

$= c_1 \cdot 0 + c_2 \cdot 0$ (using Ⓐ).

$= 0$

$\Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of Ⓐ.

<u>Example</u>: Consider $y'' + y = 0$

By inspection we see that $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are solutions of $y'' + y = 0$.

As $y'' + y = 0$ is a linear homogeneous equation of second order, it follows by principle of superposition that $c_1 \cos x + c_2 \sin x$ is also a solution of $y'' + y = 0$ for any choice of constants c_1 & c_2

For instance, $\cos x + \sin x$, $3\cos x - 2\sin x$, $2\sin x$, $5\sin x - 8\cos x$ are all solutions of $y'' + y = 0$.

<u>AN IMPORTANT REMARK</u>

① <u>LINEARITY CANNOT BE DROPPED</u>

The following examples show that principle of superposition doesn't hold generally for a non-linear equations.

17/2.1 : Show that $y = \frac{1}{x}$ is a solution of $y' + y^2 = 0$ but if $c \neq 0$ and $c \neq 1$ then $y = \frac{c}{x}$ is not a solution.

Solution: $y = \frac{1}{x} \Rightarrow y' = -\frac{1}{x^2}$

$\therefore y' + y^2 = -\frac{1}{x^2} + \frac{1}{x^2} = 0$

$\Rightarrow y = \frac{1}{x}$ is a solution. Also $y = 0$ is a solution

But if $y = \frac{c}{x}$; $c \neq 0$, $c \neq 1$ Then

$y' = -\frac{c}{x^2}$ & $y^2 = \frac{c^2}{x^2}$

$\therefore y' + y^2 = -\frac{c}{x^2} + \frac{c^2}{x^2} = \frac{c(c-1)}{x^2} \neq 0$ $\left(\because \begin{matrix} c \neq 0 \\ c \neq 1 \end{matrix}\right)$

$\therefore y = \frac{c}{x}$ is not a solution.

18/2.1 : Show that $y = x^3$ is a solution of $yy'' = 6x^4$ but if $c^2 \neq 1$ then $y = cx^3$ is not a solution.

19/2.1 : Show that $y_1 \equiv 1$ and $y_2 \equiv \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$ but

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their sum $y_1 + y_2$ is not a solution.

② HOMOGENEITY CANNOT BE DROPPED

Consider $y'' + y = e^x$ — Ⓑ

We see that $y_1 = \sin x + \frac{e^x}{2}$

$y_2 = \cos x + \frac{e^x}{2}$

are solutions of Ⓑ but

$2y_1 + 2y_2 = 2\sin x + 2\cos x + 2e^x$ is not a solution of Ⓑ.