3. $\sqrt{\frac{4}{35}}$ 4. $\sqrt{\frac{10}{3}}$ 5. $\frac{1}{3}$

6. (a) T (b) T (c) F (d) T (e) T (f) T

7.5 LEAST-SQUARES SOLUTIONS FOR INCONSISTENT SYSTEMS

Recall that a system of linear equations $Ax = b$ is inconsistent if it has no solutions. In this section, our goal is to find an approximate solution to the system $Ax = b$, even if it is inconsistent. In other words, we would like to find a vector v such that Av is as close to b as possible, i.e., $\|Av - b\| \leq \|Az - b\|$ for all $z \in \mathbb{R}^n$. This motivates the following definition.

DEFINITION Least-Squares Solution

Let $Ax = b$ be a system of linear equations, where A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. A vector $v \in \mathbb{R}^n$ is said to be a **least-squares solution** to the system $Ax = b$ if the following condition is satisfied:

$$\|Av - b\| \leq \|Az - b\| \text{ for all } z \in \mathbb{R}^n.$$

In other words, v is a least-square solution to the system $Ax = b$ if Av is the closest vector in $\mathbb{R}^m$ to b.

Remark Since calculating a norm involves finding a sum of squares, the inequality

$$\|Av - b\| \leq \|Az - b\| \text{ for all } z \in \mathbb{R}^n$$

implies that v produces the least possible value for the sum of squares of the differences in each coordinate between Az and z over all possible vectors z. Thus, it justifies the name "least-squares solution".

The following theorem (proof omitted) enables us to find a least-square solution to the linear system $Ax = b$.

THEOREM 7.14 Let $Ax = b$ be a system of linear equations, where A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. Let W be the subspace of $\mathbb{R}^m$ given by $W = \{Ax : x \in \mathbb{R}^n\}$. Let $v \in \mathbb{R}^n$. Then the following three conditions are equivalent :

(a) v is a least-square solution to the system $Ax = b$.

(b) v satisfies $(A^T A)v = A^T b$.

(c) v satisfies $Av = proj_W b$.

Note The avove theorem shows that a least-squares solution to the system $Ax = b$ can be found by solving the linear system $(A^T A)x = A^T b$.

EXAMPLE 19 Find a least-squares solution for the linear system $Ax = b$, where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$ [Delhi Univ. GE-2, 2017]

SOLUTION To find a least-squares solution for the system $Ax = b$, we need to solve the linear system $(A^TA)x = A^Tb$. Now,

$$A^TA = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix} \quad \text{and} \quad A^Tb = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

Thus, the augmented matrix for the system $(A^TA)x = A^Tb$ is $\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right]$.

We now row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{21}R_1} \left[\begin{array}{cc|c} 1 & 3/7 & 26/21 \\ 9 & 11 & 19 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 9R_1} \left[\begin{array}{cc|c} 1 & 3/7 & 26/21 \\ 0 & 50/7 & 55/7 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{7}{50}R_2} \left[\begin{array}{cc|c} 1 & 3/7 & 26/21 \\ 0 & 1 & 11/10 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{3}{7}R_2} \left[\begin{array}{cc|c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$$

Thus, $v = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} = \begin{bmatrix} 0.77 \\ 1.1 \end{bmatrix}$ is the desired least-squares solution.

Note Notice that, in the preceding example,

$$Av = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 0.77 \\ 1.1 \end{bmatrix} = \begin{bmatrix} -0.33 \\ 4.84 \\ 4.18 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = b.$$

That is, Av is very close to the vector b.

Note You can also check that $\|Av - b\| \le \|Az - b\|$ for any $z \in \mathbb{R}^2$. For example, if $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$Az - b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \|Az - b\| = 1$$

and

$$Av - b = \begin{bmatrix} -0.33 \\ -0.16 \\ 0.18 \end{bmatrix} \quad \Rightarrow \quad \|Av - b\| = 0.4085$$

$$\|Av - b\| < \|Az - b\|.$$

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EXAMPLE 20 Prove that the least-squares solution for the linear system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

satisfies $\|Av - b\| \leq \|Az - b\|$, where $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. [Delhi Univ. GE-2, 2016, 2019(Modified)]

SOLUTION To find a least-squares solution, we need to solve the linear system $(A^TA)x = A^Tb$. Now,

$$A^TA = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix} \quad \text{and} \quad A^Tb = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

Thus, the augmented matrix for the system $(A^TA)x = A^Tb$ is $\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right]$.

We now row reduce the augmented matrix to obtain $\left[\begin{array}{cc|c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$.

Thus, $v = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix}$ is the desired least-squares solution. We'll now verify that

$$\|Av - b\| \leq \|Az - b\|, \text{ where } z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Av - b = \begin{bmatrix} -1/6 \\ -1/3 \\ 1/6 \end{bmatrix} \quad \Rightarrow \quad \|Av - b\| = \frac{\sqrt{6}}{6}$$

and

$$Az - b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \|Az - b\| = 1$$

$$\therefore \qquad \|Av - b\| < \|Az - b\|.$$

The following example illustrates that the linear system $Ax = b$ may have infinitely many least-squares solutions. However, all these solutions will produce the same value for Ax.

EXAMPLE 21 Find a least-squares solution to the linear system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 3 \\ 2 & -7 & 9 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 9 \\ 8 \\ -1 \end{bmatrix}$$

SOLUTION To find a least-squares solution to $Ax = b$, we need to solve the system $(A^TA)x = A^Tb$. Now,

$$A^TA = \begin{bmatrix} 24 & -4 & 28 \\ -4 & 59 & -63 \\ 28 & -63 & 91 \end{bmatrix} \quad \text{and} \quad A^Tb = \begin{bmatrix} 48 \\ 42 \\ 6 \end{bmatrix}$$

Thus, the augmented matrix for the system $(A^TA)x = A^Tb$ is :

$$\left[\begin{array}{ccc|c} 24 & -4 & 28 & 48 \\ -4 & 59 & -63 & 42 \\ 28 & -63 & 91 & 6 \end{array}\right], \text{ which row reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 1 & 15/7 \\ 0 & 1 & -1 & 6/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row reduced echelon form matrix of the augmented matrix shows that the system $(A^TA)x = A^Tb$ has infinitely many solutions. The third variable (corresponding to the third non-pivot column) is an independent variable. Setting the third variable equal to c, the complete solution set of this system is $\left\{\left[\frac{15}{7} - c,\ \frac{6}{7} + c,\ c\right] : c \in \mathbb{R}\right\}$.

EXERCISE 7.3

1. Find a least-squares solution to the inconsistent system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Also, verify that the least-squares solution v satisfies $\|Av - b\| \leq \|Az - b\|$ for the vector $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. Find a least-squares solution to the inconsistent system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

Also, verify that the least-squares solution v satisfies $\|Av - b\| \leq \|Az - b\|$ for the vector $v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

3. Find a least-squares solution to the inconsistent system $Ax = b$, where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

ANSWERS

5. (a) T (b) T (c) F (d) T (e) T

6.7 COMPUTER GRAPHICS

In this section, we introduce the concept of homogeneous coordinates. We will show that all fundamental movements (translations, rotations, reflections and scaling) in the plane can be performed by appropriate matrix multiplications in homogeneous coordinates.

Introduction to Computer Graphics

Computer graphics is an art of drawing pictures on computer screens with the help of programming. It involves computations, creation and manipulation of data. In other words, we can say that computer graphics is a rendering tool for the generation and manipulation of images.

A computer screen consists of a collection of tiny, uniformly sized **pixels** (**picture elements** or **dots**), which are arranged in a two-dimensional grid made up of columns and rows, with a single pixel at the intersection of each row and column.

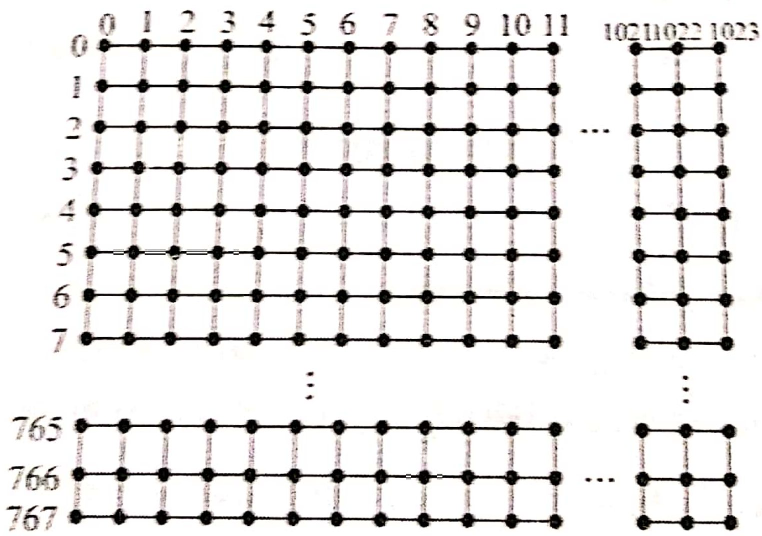

FIGURE 6.8

The number of pixels, called **resolution,** affects how much detail can be depicted in an image. Resolution is often expressed as the number of pixels in a row times number of pixels in a column.

For example, a typical 1024 × 768 computer screen would have 1024 pixels in each row (labeled '0' through "1023") and 768 pixels in each column (labeled '0' through "767") (see Figure 6.8). Pixels are normally labeled so that the y-coordinates increase as we proceed down a computer screen. In other words, the positive y-axis points "downward" instead of pointing upward. However, for practical purposes, we will continue to draw xy coordinate systems in the usual manner, with the positive y-axis pointing "upward".

The most common computer graphics technique is **raster graphics**, in which the current screen content (text, figure, icons, etc.) is stored in the memory of the computer and updated and displayed whenever a change of screen contents is necessary.

In this system, a simple two-dimensional figure with n vertices can be represented by a $2 \times n$ matrix, with each column of the matrix lists a pair of x-coordinate and y-coordinate representing a different vertex of the figure.

For example, consider the polygon in Figure 6.9 (a "Knee") with 6 vertices. We can represent this polygon algebraically by storing its 6 vertices as 6 columns in the following 2×6 matrix

$$\begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \end{bmatrix}$$

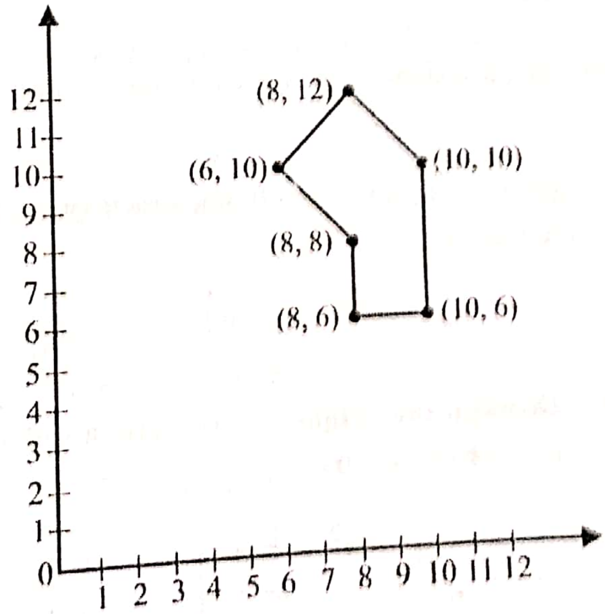

FIGURE 6.9

Whenever we move a given figure on the screen, the x-coordinate and y-coordinate of new vertices of the figure may not be integers. For simplicity, we assume that whenever a figure is manipulated (by rotation, reflection, or scaling), we round off each x- and y-coordinates to the nearest integers.

Fundamental Movements in the Plane

A **similarity** is a mapping of the plane to itself that moves every figure in the plane to another figure (the image) so that the figure and its image are similar in shape and related by the same ratio of sizes.

It can be shown that any similarity can be accomplished by composing one or more of the following mappings.

1. **Translation** : A translation is a mapping that shifts all points of a figure along a given vector.
2. **Rotation** : A rotation is a mapping that rotates all points of a figure about a fixed point (called the **center of rotation**), through a fixed angle θ. Thus, a rotation is a mapping that turns a figure around a fixed point.

Note Unless otherwise stated, all rotations are assumed to be in a *counterclockwise* direction in the plane.

3. **Reflection** : A reflection is a mapping that reflects all points of a figure about a given line. Thus, a reflection is a mapping that moves a figure by flipping it across the given line.
4. **Scaling** : A scaling is a mapping that changes the size of a figure by dilating (or contracting) the distance of all points in the figure from a given center point.

Note Each of the first three fundamental movements does not change the shape or size of a figure. In other words, it maps a given figure to a *congruent* figure. Any such map is called an **isometry**.

We now discuss how to find the new vertices of a figure, using ordinary coordinates in $\mathbb{R}^2$, that is moved by performing each of the above operations. As we shall see, all translations are straightforward, but for the last three types of operations, we shall first assume that all movements are performed "about the origin".

1. **Translation** : To perform a translation of a vertex along a vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we simply add $\begin{bmatrix} a \\ b \end{bmatrix}$ to the vertex.
2. **Rotation about the origin** : To perform a rotation of a vertex about the origin, we simply multiply on the left by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

3. **Reflection about a line through the origin** : To perform a reflection of a vertex about the line $y = mx$, we multiply on the left by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Two special cases :

(*a*) *Line of reflection is the x-axis* (*i.e.*, $m = 0$). In this case, the reflection matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(*b*) *Line of reflection is the y-axis* (*i.e.*, $m \to \infty$). In this case, the reflection matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. **Scaling from the origin** : To perform scaling about the origin with scale factors of c in the x-direction and d in the y-direction, we multiply on the left by the matrix

$$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}.$$

Note Notice that the last three types of mappings (rotation about the origin, reflection about a line through the origin, scaling about the origin) are all linear transformations and each of these can be performed using matrix multiplication. However, (non-trivial) translations are not linear transformations and cannot be performed using matrix multiplication.

EXAMPLE 51 For the polygon in Figure 6.9 (a "Knee"), use ordinary coordinates in $\mathbb{R}^2$ to find the new vertices after performing each indicated operation.

(*a*) translation along the vector [12, 6]

(*b*) rotation about the origin through $\theta = 90°$

(*c*) reflection about the line $y = -3x$

(*d*) scaling about the origin with scale factors of 1/2 in the x-direction and 4 in the y-direction.

SOLUTION (*a*) To perform a translation of a vertex along the vector [12, 6], we simply add $\begin{bmatrix} 12 \\ 6 \end{bmatrix}$ to the vertex. Thus, adding $\begin{bmatrix} 12 \\ 6 \end{bmatrix}$ to each of the vertices of the given polygon, we obtain

$\begin{bmatrix} 8 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 8 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 20 \\ 14 \end{bmatrix}$, and so on

Hence, new vertices of the polygon after performing translation are

$$(20, 12), (20, 14), (18, 16), (20, 18), (22, 16), (22, 12)$$

(*b*) A rotation of (x, y) about the origin through the angle $\theta = 90°$ can be accomplished by multiplying $\begin{bmatrix} x \\ y \end{bmatrix}$ on the left by the matrix

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For example, multiplying the vertex $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$ on the left by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ yields

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$$

Similarly, we can find other new vertices.

Note that the new vertices can also be obtained directly by performing the rotation on all vertices of the figure simultaneously.

Thus, performing the rotation on all vertices of the figure simultaneously, we obtain

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \end{bmatrix} = \begin{bmatrix} -6 & -8 & -10 & -12 & -10 & -6 \\ 8 & 8 & 6 & 8 & 10 & 10 \end{bmatrix}$$

(c) A reflection of (x, y) about the line $y = -3x$ can be obtained by multiplying $\begin{bmatrix} x \\ y \end{bmatrix}$ on the left by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{bmatrix}=\frac{1}{1+9}\begin{bmatrix}1-9 & -6\\ -6 & 9-1\end{bmatrix}=\begin{bmatrix}-4/5 & -3/5\\ -3/5 & 4/5\end{bmatrix}$$

Performing the reflection on all vertices of the polygon simultaneously, we obtain

$$\begin{bmatrix}-4/5 & -3/5\\ -3/5 & 4/5\end{bmatrix}\begin{bmatrix}8 & 8 & 6 & 8 & 10 & 10\\ 6 & 8 & 10 & 12 & 10 & 6\end{bmatrix}$$

$$\approx\begin{bmatrix}-10 & -13 & -11 & -14 & -14 & -12\\ 0 & 2 & 4 & 5 & 2 & -1\end{bmatrix}\text{ (rounding to the nearest integer)}$$

(*d*) A scaling of (x, y) about the origin by a factor of $c = 1/2$ in the x-direction and $d = 4$ in the y-direction can be accomplished by multiplying $\begin{bmatrix}x\\ y\end{bmatrix}$ on the left by the matrix $\begin{bmatrix}1/2 & 0\\ 0 & 4\end{bmatrix}$.

Performing the scaling on all vertices of the polygon simultaneously, we obtain

$$\begin{bmatrix}1/2 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}8 & 8 & 6 & 8 & 10 & 10\\ 6 & 8 & 10 & 12 & 10 & 6\end{bmatrix}=\begin{bmatrix}4 & 4 & 3 & 4 & 5 & 5\\ 24 & 32 & 40 & 48 & 40 & 24\end{bmatrix}.$$

Homogeneous Coordinates

We have already seen that the following three mappings

(*a*) rotation about the origin,

(*b*) reflection about the line through the origin, and

(*c*) scaling with the origin as center

are all linear transformations and can all be performed using matrix multiplications. Unfortunately, (non-trivial) translations are not linear transformations and they can not be performed using matrix multiplication. Our purpose now is to represent all these transformations in a consistent way so that they can be combined easily. This can be achieved by introducing a different type of coordinate system taken from projective geometry, called **homogeneous coordinates.**

In homogeneous coordinates, we add a third coordinate to a point. Instead of being represented by a two-dimensional point (x, y), each point is represented by a three-dimensional point $(x, y, 1)$. We define any three-dimensional point of the form $(tx, ty, t) = t(x, y, 1)$, where $t \neq 0$, to be **equivalent** to the ordinary two-dimensional point (x, y). Thus, for each two-dimensional point (x, y), there is an infinite set of homogeneous coordinates (tx, ty, t), $(t \neq 0)$ in three-dimensional that are all equivalent to (x, y). For example, the points (2, 3, 1), (4, 6, 2) = 2(2, 3, 1) and (6, 9, 3) = 3(2, 3, 1) are all equivalent to (2, 3). A point in homogeneous coordinates is said to be in **normalized form** if its last coordinate is 1. Notice that any point in homogeneous coordinates can be normalized simply by dividing all three coordinates of a triple by its last coordinate. Thus, each two-dimensional point has a unique set of normalized homogeneous coordinates, which is said to be its **standard form**. For example, the points (15, –9, 6), (10, –6, 4) and (5/2, –3/2, 1) are all homogeneous coordinates for the two-dimensional point (5/2, –3/2), in which its standard form is (5/2, –3/2, 1).

Representing Movements with Matrix Multiplication in Homogeneous Coordinates

Translation : To translate vertex (x, y) along a given vector $[a, b]$, we first replace (x, y) with its equivalent vector $[x, y, 1]$ in homogeneous coordinates. Then, we multiply on the left by the matrix

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

This gives

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix},$$

which is equivalent to the two-dimensional point $(x + a, y + b)$, as desired.

Rotation : To rotate vertex (x, y) about the origin through angle θ, we multiply its equivalent vector $[x, y, 1]$ in homogeneous coordinates on the left by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix},$$

which is equivalent to the two-dimensional point $(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$, as desired.

Reflecion : A reflection of (x, y) about the line $y = mx$ can be accomplished by multiplying $[x, y, 1]$ on the left by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & 0 \\ 0 & 0 & 1+m^2 \end{bmatrix}$$

In particular, a reflection about the y-axis can be accomplished by multiplying on the left by the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling : A scaling of (x, y) about the origin by a factor of c in the x-direction and d in the y-direction can be accomplished by multiplying $[x, y, 1]$ on the left by the matrix

$$\begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note Notice that multiplying a 3×3 matrix A by any two vectors of the form $t[x, y, 1] = [tx, ty, t]$ $[t \neq 0]$ equivalent to (x, y) always produces two results that are equivalent in homogeneous coordinates. This follows from the fact that any matrix A and any vector v (of compatible size), we have $A(tv) = t(Av)$, for any scalar t.

Movements Not Centered at the Origin : Similarity Method

We now determine the matrices for rotations, reflections, and scaling that are not centered about the origin using a method, called the **Similarity Method.** This method involves combining appropriate translation matrices with the matrices for origin-centered rotations, reflections and scaling.

Similarity Method

Step 1. Use a translation to move the figure so that the rotation, reflection or scaling to be performed is "about the origin". This means for rotation or scaling about a point $(r, s) \neq (0, 0)$, we first apply the translation that takes (r, s) to $(0, 0)$, and for reflection about the line $y = mx + b$, we vertically translate the plane down b units so that the line of reflection goes through the origin.

Step 2. Perform the desired rotation, reflection, or scaling "about the origin".

Step 3. Apply the reverse translation to get the altered figure back to the original figure.

Note Notice that the Similarity Method requires the composition of three movements and hence it can be accomplished by taking the product of the corresponding matrices for the individual mappings in *reverse* order (Theorem 6.8)

For example, a rotation about center $(r, s) \neq (0, 0)$ through angle θ can be accomplished by the matrix product

$$\underbrace{\begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (0,\,0)\text{ back to }(r,s)}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{rotate about}(0,\,0) \\ \text{through angle }\theta}} \underbrace{\begin{bmatrix} 1 & 0 & -r \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{translate from} \\ (r,\,s)\text{ to }(0,\,0)}}$$

A similarity can be performed simultaneously on multiple points by multiplying the matrix for the similarity by a matrix whose columns represent the normalized homogeneous coordinates for each point.

EXAMPLE 52 For the polygon in Figure 6.9 (a "Knee"), use homogeneous coordinates to find the new vertices after performing a rotation about the point $(r, s) = (12, 6)$ through an angle of $\theta = 90°$. Then sketch the final figure that would result from this movement.

SOLUTION To find new vertices, we first replace each (x, y) with its equivalent vector $[x, y, 1]$ in homogeneous coordinates and then follow the three steps of the Similarity Method.

Step 1. We first apply the translation that takes (12, 6) to (0, 0) in order to establish the origin as center. The matrix for this operation is

$$\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2. The second step is to perform the rotation through angle 90° about the origin. The matrix for this operation is

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3. Finally, we apply the reverse translation that takes (0, 0) back to (12, 6). The matrix for this operation is

$$\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

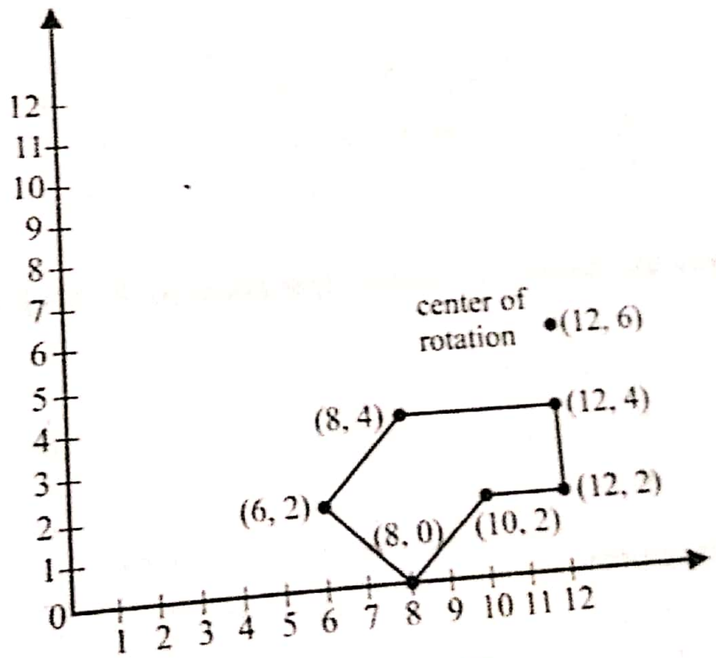

FIGURE 6.10

The combined result of these operations is

$$\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 18 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the rotation on all vertices of the figure simultaneously, we obtain

$$\begin{bmatrix} 0 & -1 & 18 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 10 & 8 & 6 & 8 & 12 \\ 2 & 2 & 0 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The columns of the final matrix (ignoring the last row entries) give the vertices of the rotated figure, as shown in Figure 6.10.

EXAMPLE 53 Use homogeneous coordinates to find new vertices of the "Knee" (Refer to Figure 6.9) after performing a reflection about the line $y = -3x + 30$. Then sketch the final figure that would result from this movement.

SOLUTION As before, we first replace (x, y) with its equivalent vector $[x, y, 1]$, and follow the *Similarity Method.*

Step 1. The first step is to apply the translation so that the line $y = -3x + 30$ goes through the origin. This can be achieved by performing the translation that takes $(0, 30)$ to $(0, 0)$. The matrix for this operation is :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -30 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2. We next perform a reflection about the corresponding line $y = -3x$. The matrix for this operation is

$$\frac{1}{1+(-3)^2}\begin{bmatrix} 1-(-3)^2 & 2(-3) & 0 \\ 2(-3) & (-3)^2-1 & 0 \\ 0 & 0 & 1+(-3)^2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -8 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Step 3. Finally, we apply the reverse translation that takes $(0, 0)$ back to $(0, 30)$. The matrix for this operation is :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix}$$

The combined result of these operations is

$$\frac{1}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -8 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -30 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -8 & -6 & 180 \\ -6 & 8 & 60 \\ 0 & 0 & 10 \end{bmatrix}$$

Performing the reflection on all vertices of the figure simultaneously, we obtain

$$\frac{1}{10}\begin{bmatrix} -8 & -6 & 180 \\ -6 & 8 & 60 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 8 & 7 & 7 & 4 & 4 & 6 \\ 6 & 8 & 10 & 8 & 8 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

after rounding the results for each vertex to the nearest integer. The columns of the final matrix (ignoring the last entries) give the vertices of the reflected figure, as shown in Figure 6.11. Notice that the figure is slightly distorted because of the rounding involved.

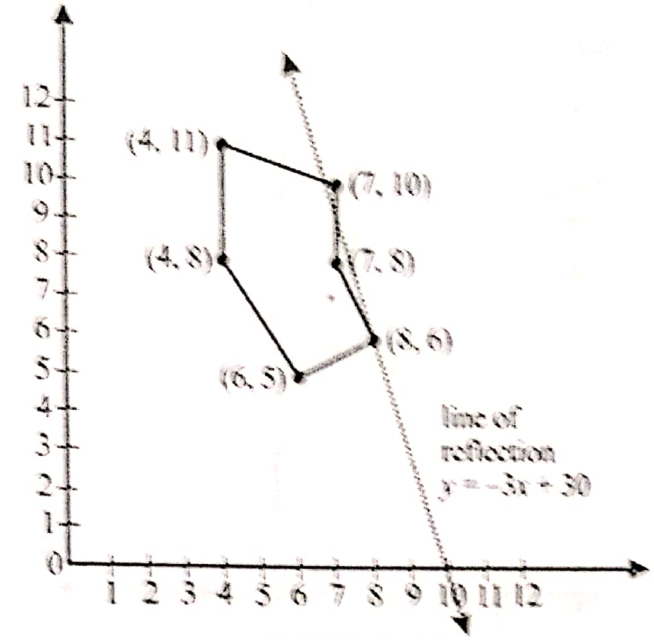

FIGURE 6.11

EXAMPLE 54 Use homogeneous coordinates to find the new vertices of the "Knee" (Refer to Figure 6.9) after performing scaling about (6, 10) with a factor of $c = 1/2$ in the x-direction and $d = 4$ in the y-direction.

SOLUTION Following the procedure similar to Examples 50 and 51, it can be checked that the matrix for the desired scaling is the product of the following three matrices :

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

This reduces to $\begin{bmatrix} 1/2 & 0 & 3 \\ 0 & 4 & -30 \\ 0 & 0 & 1 \end{bmatrix}$.

Scaling all vertices of the figure simultaneously, we obtain

$$\begin{bmatrix} 1/2 & 0 & 3 \\ 0 & 4 & -30 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 8 & 6 & 8 & 10 & 10 \\ 6 & 8 & 10 & 12 & 10 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 6 & 7 & 8 & 8 \\ -6 & 2 & 10 & 18 & 10 & -6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The columns of the final matrix (ignoring the last row entries) give the vertices of the scaled figure, as shown in Figure 6.12. Notice that two of the scaled vertices have negative y-coordinates, and hence would not be displayed on the computer screen.

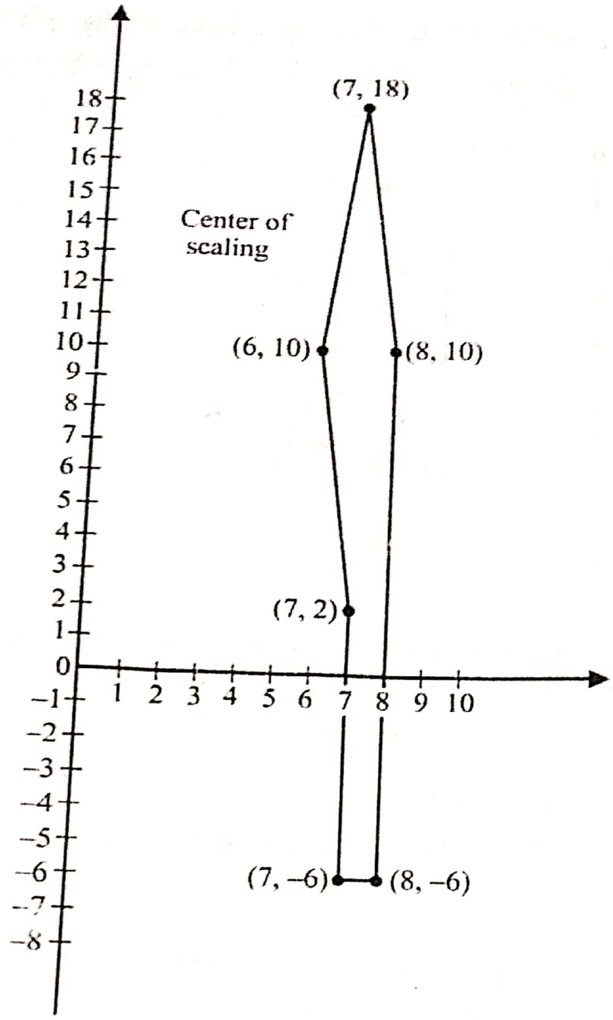

FIGURE 6.12

EXAMPLE 55 For the graphic figure below, use homogeneous coordinates to find the new vertices after performing a scaling about the point (3, 3) with scale factors of 3 in the x-direction and 2 in the y-direction. Then sketch the final figure that would result from this movement :

[*Delhi Univ. GE*-2, 2019]

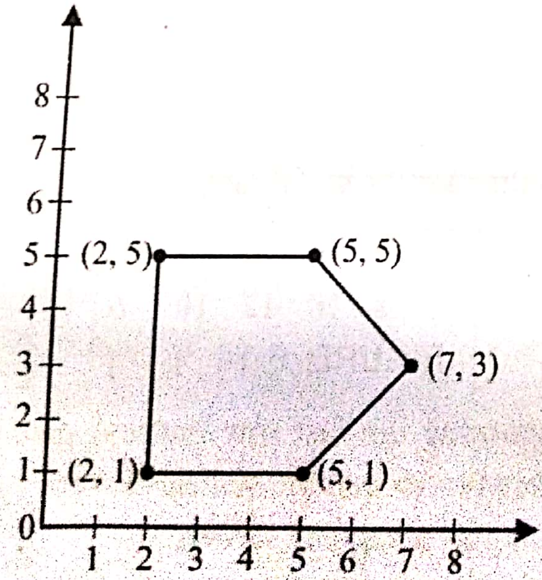

FIGURE 6.13

SOLUTION To find new vertices, we first replace each (x, y) with its equivalent vector $[x, y, 1]$ in homogeneous coordinates and then follow the three steps of the Similarity Method.

Step 1 We first apply the translation that takes (3, 3) to (0, 0) in order to establish the origin as the center. The matrix for this operation is

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2 The second step is to perform a scaling about (0, 0) with scale factors of 3 in the x-direction and 2 in the y-direction. The matrix for this operation is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3 Finally, we apply the reverse translation that takes (0, 0) back to (3, 3). The matrix for this operation is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

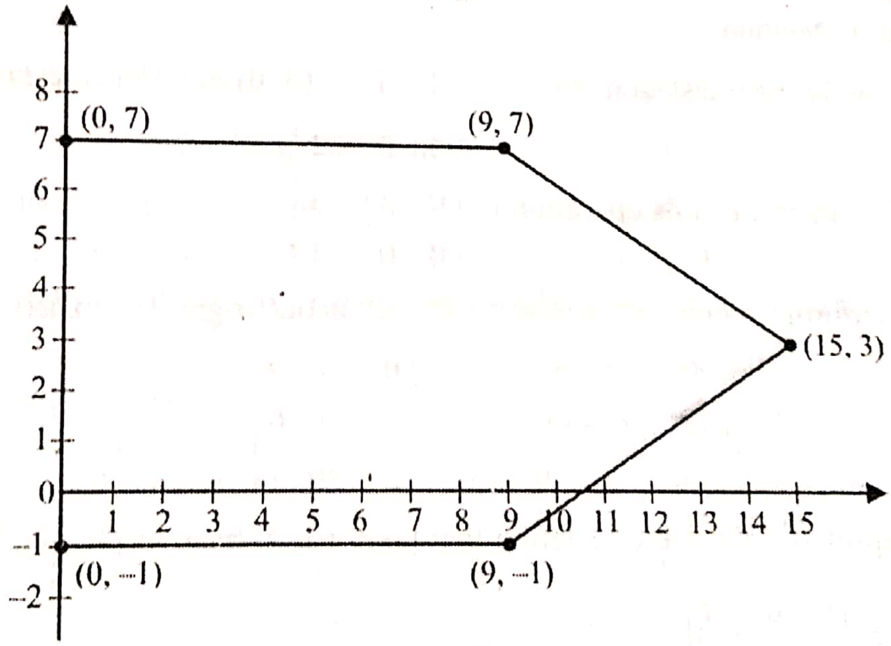

FIGURE 6.14

The combined result of these operations is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the operation of scaling on all vectors of the figure simultaneously, we get

$$\begin{bmatrix} 3 & 0 & -6 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 5 & 7 & 5 \\ 1 & 5 & 5 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 9 & 15 & 9 \\ -1 & 7 & 7 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The columns of the final matrix(ignoring the last rows entries) give the vertices of the scaled figure, as shown in Figure 6.14. Two of the scaled vertices have negative y coordinates, and hence would not be displayed on the computer screen.

Composition of Movements

Since every movement in homogeneous coordinates can be represented by a matrix multiplication, the composition of movements can also be represented by a matrix multiplication in homogeneous coordinates.

EXAMPLE 56 Use the similarity method to show that a rotation about the point $(1, -1)$ through an angle $\theta = 90°$, followed by a reflection about the line $x = 1$ is represented by the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

[*Delhi Univ. GE-2, 2017*]

SOLUTION We first replace each (x, y) with its vector $[x, y, 1]$ in homogeneous coordinates and follow the Similarity Method.

Step 1. We first apply the translation that takes $(1, -1)$ to $(0, 0)$ in order to establish the origin as center. The matrix of this operation is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Step 2. We next perform a rotation through angle $\theta = 90°$ about origin. The matrix of this operation is

$$\begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3. We then apply the reverse translation that takes $(0, 0)$ back to $(1, -1)$. The matrix of this operation is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

So, the net effect of these 3 operations is to rotate each vertex about $(1, -1)$ through an angle $\theta = 90°$. The combined result of these operations is :

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, observe that the line $x = 1$ is parallel to y-axis which passes through (1, 0). As before, we follow the Similarity Method.

Step 1. We translate from (1, 0) to (0, 0) so that the line passes through the origin. The matrix of this operation is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 2. We perform a reflection about the line $x = 0$ (*i.e.*, y-axis). The matrix of this operation is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3. We translate from (0, 0) back to (1, 0). The matrix of this operation is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The combined result of these operations is :

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the matrix for the given composition is the product of the following six matrices:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 57 For the polygon in Figure 6.9 (a "Knee"), use homogeneous coordinates to find the new vertices after performing the following sequence of operations :

A rotation about (8, 10) through $\theta = 300°$, followed by a reflection about the line $y = (-1/2)x + 20$

SOLUTION The matrix for the indicated rotation is

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos 300° & -\sin 300° & 0 \\ \sin 300° & \cos 300° & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix for the indicated reflection is

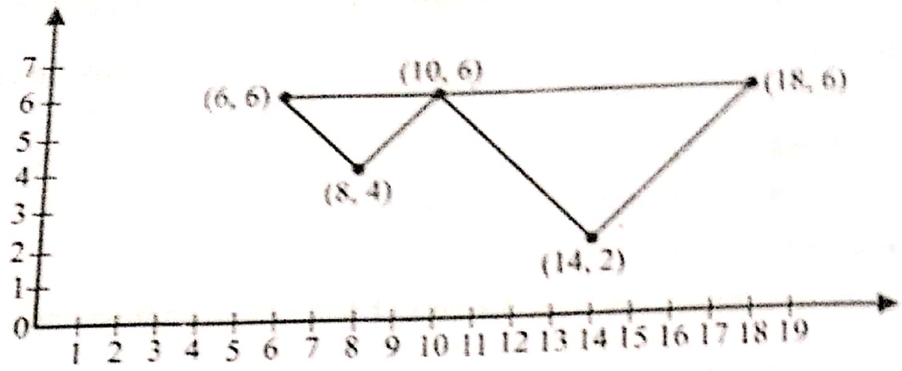

FIGURE 6.17

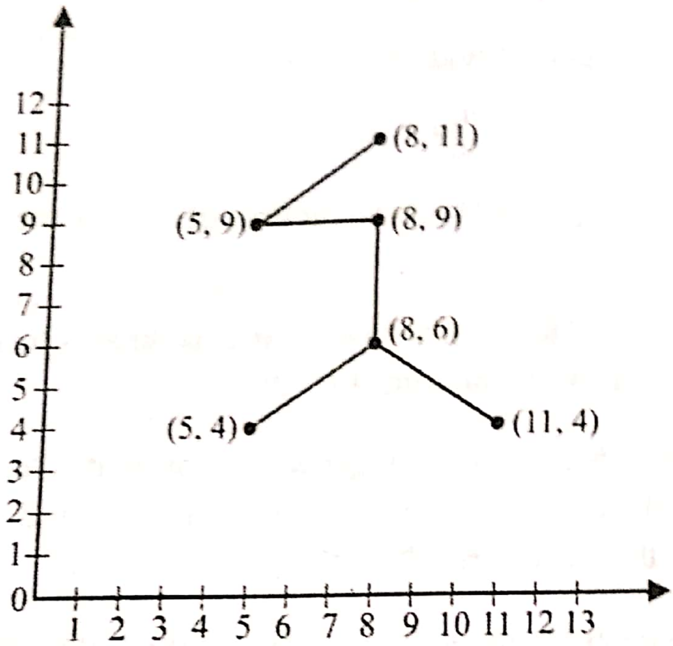

FIGURE 6.18

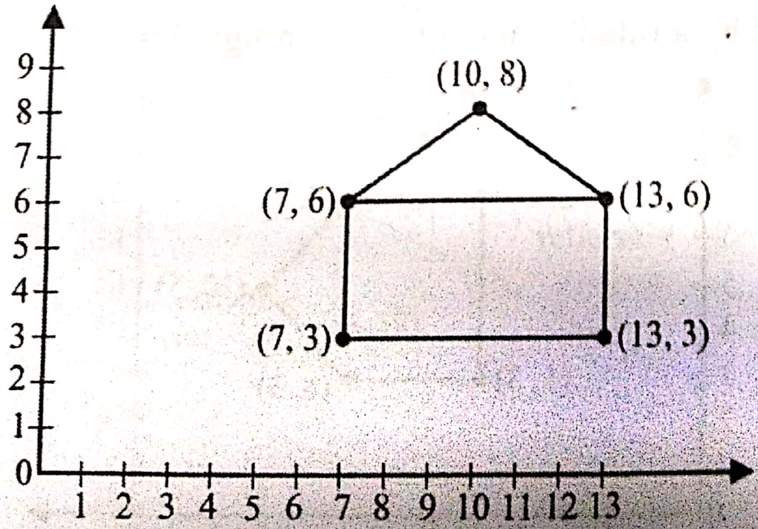

FIGURE 6.19

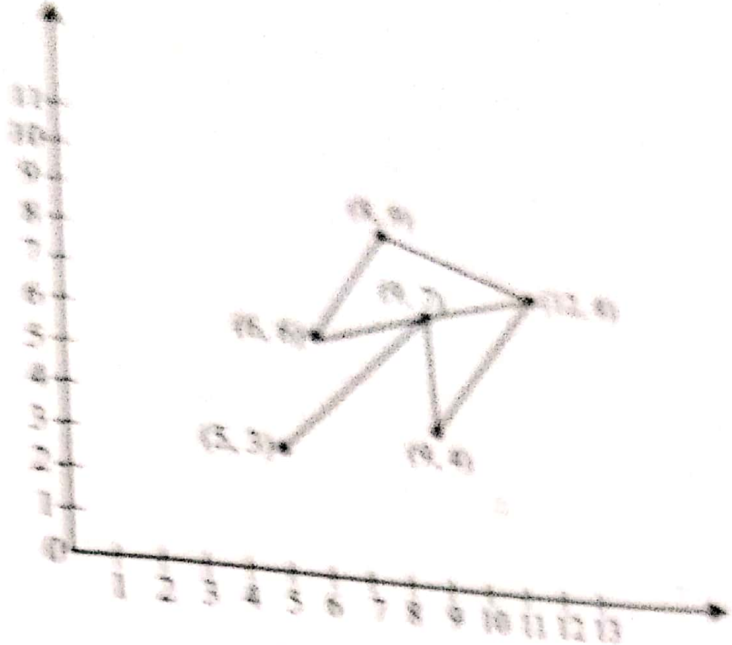

FIGURE 6.20

ANSWERS

1. (*a*) (9, 1), (9, 5), (12, 1), (12, 5), (14, 3)
   (*b*) (3, 5), (1, 9), (5, 7), (3, 10), (16, 9)
   (*c*) (–2, 5), (0, 9), (–5, 7), (–2, 10), (–5, 10)
   (*d*) (20, 6), (20, 14), (32, 6), (32, 14), (40, 10)

2. (*a*) (3, 11), (5, 9), (7, 11), (11, 7), (15, 11)
   (*b*) (–8, 2), (–7, 5), (–10, 6), (–9, 11), (–14, 13)
   (*c*) (8, 1), (8, 4), (11, 4), (10, 10), (16, 11)
   (*d*) (3, 18), (4, 12), (5, 18), (7, 6), (9, 18)

3. (14, 9), (10, 6), (11, 11), (8, 9), (6, 8), (11, 14)
4. (2, 4), (8, 5), (8, 11), (2, 11), (8, 6), (14, 4)
5. (0, 5), (1, 7), (0, 11), (–5, 8), (–4, 10)
6. (1, 18), (–3, 13), (–6, 8), (–2, 17), (–5, 12), (–6, 10).