Series of functions

Definition: If $\langle f_n \rangle$ is a sequence of functions defined on a subset D of R, with values in R; the sequence of partial sums (S.O.P.S.) $\langle s_n \rangle$ of the infinite series Σf_n is defined for all x in D by

$$s_1(x) = f_1(x)$$

$$s_2(x) = f_1(x) + f_2(x) = s_1(x) + f_2(x)$$

$$s_3(x) = f_1(x) + f_2(x) + f_3(x) = s_2(x) + f_3(x)$$

$$\vdots$$

$$s_{n+1}(x) = s_n(x) + f_{n+1}(x).$$

In case the sequence $\langle s_n \rangle$ of functions converges on D to a function f, we say that Σf_n converges to f uniformly on D, or Σf_n is uniformly summable to f on D.

Scanned with CamScanner

Theorem 1B: If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and $\sum f_n$ converges to f uniformly on D, then f is continuous on D.

Pf: The proof follows as a direct translation of Theorem 1A.

Theorem 2B: Cauchy's criterion for uniform convergence of series of functions.

Let $\langle f_n \rangle$ be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniform convergent on D iff for each $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \quad \forall\, x \in D$$
$$n, m \geq n_0.$$

Pf: The proof follows immediately from Cauchy's convergence criterion for uniform convergence of sequences of functions. (applied to S.O.P.S.)

Theorem 3 B : Weierstrass M- Test

Let $\langle M_n \rangle$ be a sequence of positive real numbers such that $\left. |f_n(x)| \leq M_n \right\} \forall x \in D, n \in \mathbb{N}$.

If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof: Let $\epsilon > 0$

As $\sum M_n$ is convergent, corresponding $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t

$$|M_{n+1} + M_{n+2} + \cdots + M_m| < \epsilon \quad \forall\ n, m \geq n_0$$

Hence, for all $x \in D$, $n, m \geq n_0$ we have

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)|$$

$$\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_m(x)|$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_m$$

$$< \epsilon$$

$\Rightarrow$ $\sum f_n$ is uniformly convergent on D (by Theorem 2 B)

Examples. ① The series $\sum r^n \cos n\theta$, $\sum r^n \sin n\theta$, $\sum r^n \cos n^2\theta$, $\sum r^n \sin(a^n\theta)$ are all unif. convergent for all $\theta \in \mathbb{R}$ $(0<r<1)$.

[Taking $M_n = r^n$].

② The series $\sum a_n \left(\frac{x^n}{1+x^{2n}}\right)$ converges uniformly for all real x, if $\sum a_n$ is absolutely convergent.

③ $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$ is unif cgt for all real x

[Take $M_n = \frac{1}{n(n+1)}$]

④ $\sum \frac{\cos n\theta}{n^p}$ is unif cgt $\forall$ real θ, $p>1$

[Take $M_n = \frac{1}{n^p}$].

⑤ $\sum n^{-x}$ is unif cgt in $[1+\delta, \infty)$, $\delta>0$.

$\left[f_n(x) = \frac{1}{n^x} \leq \frac{1}{n^{1+\delta}}\right]$ $\left(\begin{array}{l} x \geq 1+\delta \\ n^x \geq n^{1+\delta} \end{array}\right)$

Scanned with CamScanner

(6) Show that each of the series $\sum \frac{1}{n^4+n^2x^2}$, $\sum \frac{1}{n^2+n^4x^2}$ is uniformly convergent on $\mathbb{R}$.

$$f_n(x) = \frac{1}{n^2(n^2+x^2)} \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R}.$$

$\therefore$ take $M_n = \frac{1}{n^2}$

As $\sum \frac{1}{n^2}$ is convergent, $\sum \frac{1}{n^2(n^2+x^2)}$ is u.c. on $\mathbb{R}$.

(7) $\sum \frac{x^n}{n^2}$ is u.c. on $[-1,1]$

(8) $\sum \frac{x^n}{n(n+1)}$ is u.c. on $[-1,1]$.

Scanned with CamScanner