Interchange of Limits

Examples : ① Let $g_n(x) = x^n$, $x \in [0,1]$ & $n \in \mathbb{N}$.
This sequence of functions $\langle g_n \rangle$ converges pointwise to the function $g(x)$ defined as:

$$g(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Although all the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$.

② Each of the functions $g_n(x) = x^n$ in above example has a continuous derivative on $[0,1]$. However, the limit function g does not have a derivative at $x = 1$ since it is not continuous at that point.

③ Let $f_n : [0,1] \to \mathbb{R}$ be defined as :-

$$f_n(x) = \left.\begin{cases} n^2 x, & 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right), & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1 \end{cases}\right\} \forall n \geq 2$$

Note that each function f_n is continuous on $[0,1]$ and hence R-integrable on $[0,1]$.

Also $\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{n}} n^2 x\,dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (-n^2x + 2n)\,dx$

$+ \int_{\frac{2}{n}}^{1} 0\,dx$

$= n^2\left[\frac{x^2}{2}\right]_0^{\frac{1}{n}} + \left[-\frac{n^2x^2}{2} + 2nx\right]_{\frac{1}{n}}^{\frac{2}{n}}$

$= \frac{1}{2} - 2 + 4 + \frac{1}{2} - 2$

$= 1$

Now the pointwise limit function of $\langle f_n(x)\rangle$ is given by $f(x) = 0 \ \forall x \in [0,1]$.

Now $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} (1) = 1$

$\neq 0$

$= \int_0^1 \lim_{n\to\infty} f_n(x)$

Thus, limit of integrals is not the same as integral of limits in this case.

(4). Define $\langle f_n \rangle$ on $[0,1]$ as

$$f_n(x) = 2nx\,e^{-nx^2} \quad \forall\ n = 1, 2, \dots$$

$\langle f_n \rangle \to f$ pointwise on $[0,1]$ where
$f(x) = 0 \quad \forall\ x \in [0,1]$

Now $$\int_0^1 f_n(x)\,dx = \int_0^1 2nx\,e^{-nx^2}$$

$$= \left[-e^{-nx^2}\right]_0^1 = 1 - e^{-n}.$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} (1 - e^{-n}) = 1$$

but $$\int_0^1 \lim_{n\to\infty} f_n(x) = \int_0^1 0\,dx = 0.$$

$\therefore$ in this case also, lt. of integrals is not the same as integral of limits.

Theorem 1A: UNIFORM LIMIT OF A SEQUENCE OF CONTINUOUS FUNCTIONS IS CONTINUOUS

Let $\langle f_n \rangle$ be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that $\langle f_n \rangle \to f$ uniformly on A. Then f is continuous on A.

Pf:- Let $\langle f_n \rangle$ be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$.

Let $\langle f_n \rangle \to f$ unif. on A.

(To show: f is continuous on A)

Let $c \in A$ be arbitrary but fixed.

(To show: f is continuous at c).

Let $\epsilon > 0$ be given.

(T.S $\exists\ \delta > 0$ s.t $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$)

As $\langle f_n \rangle \to f$ unif on A; corresponding to $\frac{\epsilon}{3} > 0$, $\exists\ \delta > 0$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \forall\ n \geq n_0,\ x \in A.$$

In particular $|f_n(c) - f(c)| < \frac{\epsilon}{3} \quad \forall\ n \geq n_0.$ —(1)

Choose $m \geq n_0$.

As f_m is continuous at $x = c$, corresp. to $\frac{\epsilon}{3} > 0$, $\exists$ $\delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f_m(x) - f_m(c)| < \frac{\epsilon}{3} \quad -(2)$$

Consider $|f(x) - f(c)|$

$$= |f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c)|$$

$$\leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{whenever } |x - c| < \delta$$

(using (1) + (2))

$\Rightarrow$ $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$

$\Rightarrow$ f is continuous at c.

As $c \in A$ was arbitrary; it follows that f is continuous on A.

Remark: Although the uniform convergence of a sequence of continuous functions is sufficient to guarantee the continuity of the limit functions, it is not necessary.

Eg $\langle f_n(x) \rangle = \langle \frac{x}{n} \rangle$ on $\mathbb{R}$
$n = 1, 2, \ldots$

Then

$\langle f_n \rangle \to 0$ pointwise on $\mathbb{R}$

Each f_n is continuous on $\mathbb{R}$, f is also continuous on $\mathbb{R}$ but the convergence is non-uniform.