7.7 MAGNETIC SCALAR AND VECTOR POTENTIALS

We recall that some electrostatic field problems were simplified by relating the electric potential V to the electric field intensity $\mathbf{E}$ ($\mathbf{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field $\mathbf{B}$. In fact, the magnetic potential could be scalar V_m or vector $\mathbf{A}$. To define V_m and $\mathbf{A}$ involves recalling two important identities (see Example 3.10 and Practice Exercise 3.10):

$$\nabla \times (\nabla V) = 0 \tag{7.35a}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{7.35b}$$

which must always hold for any scalar field V and vector field $\mathbf{A}$.

Just as $\mathbf{E} = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to $\mathbf{H}$ according to

$$\boxed{\mathbf{H} = -\nabla V_m} \quad \text{if } \mathbf{J} = 0 \tag{7.36}$$

The condition attached to this equation is important and will be explained. Combining eq. (7.36) and eq. (7.19) gives

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = 0 \tag{7.37}$$

since V_m must satisfy the condition in eq. (7.35a). Thus the magnetic scalar potential V_m is only defined in a region where $\mathbf{J} = 0$ as in eq. (7.36). We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields; hence,

$$\nabla^2 V_m = 0, \qquad (\mathbf{J} = 0) \tag{7.38}$$

We know that for a magnetostatic field, $\nabla \cdot \mathbf{B} = 0$ as stated in eq. (7.34). To satisfy eqs. (7.34) and (7.35b) simultaneously, we can define the *vector magnetic potential* $\mathbf{A}$ (in Wb/m) such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{7.39}$$

Just as we defined

$$V = \int \frac{dQ}{4\pi\varepsilon_o r} \tag{7.40}$$

we can define

$$\boxed{\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}}{4\pi R}} \quad \text{for line current} \tag{7.41}$$

$$\mathbf{A} = \int_S \frac{\mu_o \mathbf{K}\, dS}{4\pi R} \quad \text{for surface current} \tag{7.42}$$

$$\mathbf{A} = \int_v \frac{\mu_o \mathbf{J}\, dv}{4\pi R} \quad \text{for volume current} \tag{7.43}$$

Rather than obtaining eqs. (7.41) to (7.43) from eq. (7.40), an alternative approach would be to obtain eqs. (7.41) to (7.43) from eqs. (7.6) to (7.8). For example, we can derive eq. (7.41) from eq. (7.6) in conjunction with eq. (7.39). To do this, we write eq. (7.6) as

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I\, d\mathbf{l}' \times \mathbf{R}}{R^3} \tag{7.44}$$

where $\mathbf{R}$ is the distance vector from the line element $d\mathbf{l}'$ at the source point (x', y', z') to the field point (x, y, z) as shown in Figure 7.19 and $R = |\mathbf{R}|$, that is,

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \tag{7.45}$$

Hence,

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{R}}{R^3}$$

or

$$\frac{\mathbf{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \quad \left(= \frac{\mathbf{a}_R}{R^2}\right) \tag{7.46}$$

where the differentiation is with respect to x, y, and z. Substituting this into eq. (7.44), we obtain

$$\mathbf{B} = -\frac{\mu_o}{4\pi} \int_L I\, d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) \tag{7.47}$$

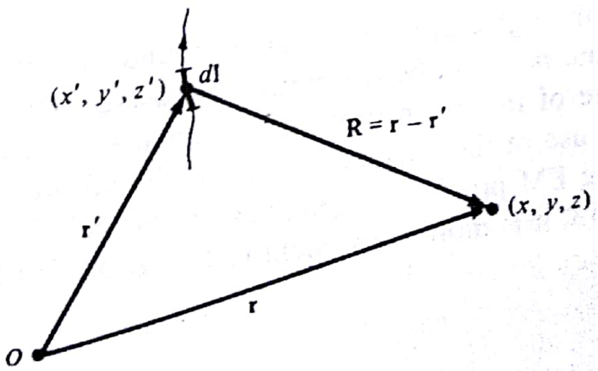


Figure 7.19 Illustration of the source point (x', y', z') and the field point (x, y, z).

We apply the vector identity

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F} \tag{7.48}$$

where f is a scalar field and $\mathbf{F}$ is a vector field. Taking $f = 1/R$ and $\mathbf{F} = d\mathbf{l}'$, we have

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{l}' - \nabla \times \left(\frac{d\mathbf{l}'}{R}\right)$$

Since ∇ operates with respect to (x, y, z) while $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = 0$. Hence,

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\mathbf{l}'}{R} \tag{7.49}$$

With this equation, eq. (7.47) reduces to

$$\mathbf{B} = \nabla \times \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R} \tag{7.50}$$

Comparing eq. (7.50) with eq. (7.39) shows that

$$\mathbf{A} = \int_L \frac{\mu_o I\, d\mathbf{l}'}{4\pi R}$$

verifying eq. (7.41).

By substituting eq. (7.39) into eq. (7.32) and applying Stokes's theorem, we obtain

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

or

$$\boxed{\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l}} \tag{7.51}$$

Thus the magnetic flux through a given area can be found by using either eq. (7.32) or (7.51). Also, the magnetic field can be determined by using either V_m or $\mathbf{A}$; the choice is dictated by the nature of the given problem except that V_m can be used only in a source-free region. The use of the magnetic vector potential provides a powerful, elegant approach to solving EM problems, particularly those relating to antennas. As we shall notice in Chapter 13, it is more convenient to find $\mathbf{B}$ by first finding $\mathbf{A}$ in antenna problems.

8.3 MAGNETIC TORQUE AND MOMENT

Now that we have considered the force on a current loop in a magnetic field, we can determine the torque on it. The concept of a current loop experiencing a torque in a magnetic field is of paramount importance in understanding the behavior of orbiting charged particles, dc motors, and generators. If the loop is placed parallel to a magnetic field, it experiences a force that tends to rotate it.

> The **torque T** (or mechanical moment of force) on the loop is the vector product of the force **F** and the moment arm **r**.

That is,

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{8.14}$$

and its units are newton-meters (N · m).

Let us apply this to a rectangular loop of length ℓ and width w placed in a uniform magnetic field **B** as shown in Figure 8.5(a). From this figure, we notice that $d\mathbf{l}$ is parallel to **B** along sides AB and CD of the loop and no force is exerted on those sides. Thus

$$\mathbf{F} = I\int_{B}^{C} d\mathbf{l} \times \mathbf{B} + I\int_{D}^{A} d\mathbf{l} \times \mathbf{B}$$
$$= I\int_{0}^{\ell} dz\, \mathbf{a}_z \times \mathbf{B} + I\int_{\ell}^{0} dz\, \mathbf{a}_z \times \mathbf{B}$$

or

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0 \tag{8.15}$$

where $|\mathbf{F}_o| = IB\ell$ because **B** is uniform. Thus, no force is exerted on the loop as a whole. However, $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points on the loop, thereby creating a couple. If the

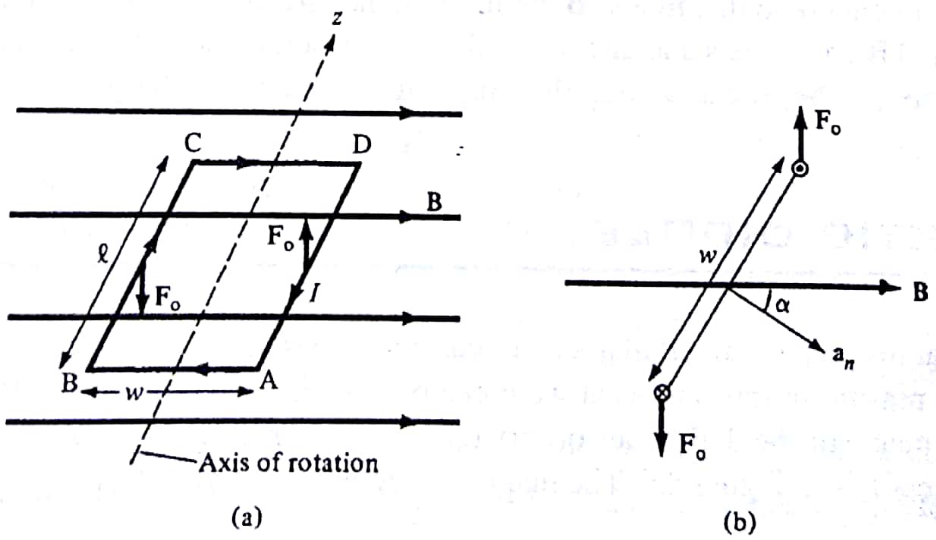


Figure 8.5 **(a)** Rectangular planar loop in a uniform magnetic field. **(b)** Cross-sectional view of part (a).

normal to the plane of the loop makes an angle α with $\mathbf{B}$, as shown in the cross-sectional view of Figure 8.5(b), the torque on the loop is

$$|\mathbf{T}| = |\mathbf{F}_o| \, w \sin \alpha$$

or

$$T = BI\ell w \sin \alpha \tag{8.16}$$

But $\ell w = S$, the area of the loop. Hence,

$$T = BIS \sin \alpha \tag{8.17}$$

We define the quantity

$$\boxed{\mathbf{m} = IS\mathbf{a}_n} \tag{8.18}$$

as the *magnetic dipole moment* (in $A \cdot m^2$) of the loop. In eq. (8.18), $\mathbf{a}_n$ is a unit normal vector to the plane of the loop and its direction is determined by the right-hand rule: fingers in the direction of current and thumb along $\mathbf{a}_n$.

The **magnetic dipole moment** is the product of current and area of the loop; its direction is normal to the loop.

Introducing eq. (8.18) in eq. (8.17), we obtain

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B}} \tag{8.19}$$

Although this expression was obtained by using a rectangular loop, it is generally applicable in determining the torque on a planar loop of any arbitrary shape. The only limitation is that the magnetic field must be uniform. It should be noted that the torque is in the direction of the axis of rotation (the z-axis in the case of Figure 8.5a). It is directed with the aim of reducing α so that $\mathbf{m}$ and $\mathbf{B}$ are in the same direction. In an equilibrium position (when $\mathbf{m}$ and $\mathbf{B}$ are in the same direction), the loop is perpendicular to the magnetic field and the torque will be zero as well as the sum of the forces on the loop.

8.4 A MAGNETIC DIPOLE

A bar magnet or a small filamentary current loop is usually referred to as a *magnetic dipole*. The reason for this and what we mean by "small" will soon be evident. Let us determine the magnetic field $\mathbf{B}$ at an observation point $P(r, \theta, \phi)$ due to a circular loop carrying current I as in Figure 8.6. The magnetic vector potential at P is

$$\mathbf{A} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{l}}{r} \tag{8.20}$$

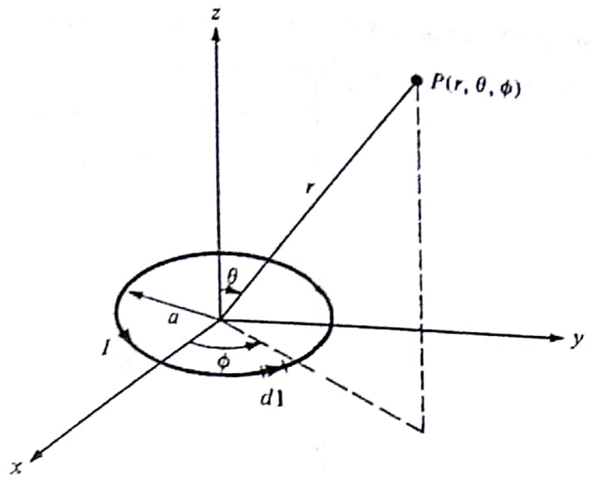


Figure 8.6 Magnetic field at P due to a current loop.

It can be shown that in the far field $r \gg a$, so that the loop appears small at the observation point, $\mathbf{A}$ has only ϕ-component and it is given by

$$\mathbf{A} = \frac{\mu_o I \pi a^2 \sin\theta \, \mathbf{a}_\phi}{4\pi r^2} \tag{8.21a}$$

or

$$\boxed{\mathbf{A} = \frac{\mu_o \, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2}} \tag{8.21b}$$

where $\mathbf{m} = I\pi a^2 \mathbf{a}_z$, the magnetic moment of the loop, and $\mathbf{a}_z \times \mathbf{a}_r = \sin\theta \, \mathbf{a}_\phi$. We determine the magnetic flux density $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$ as

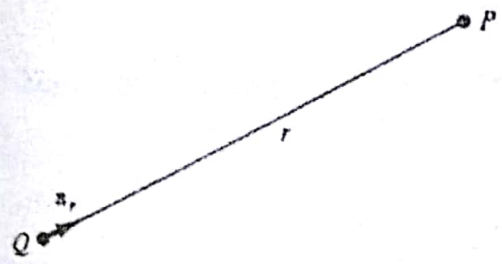
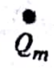
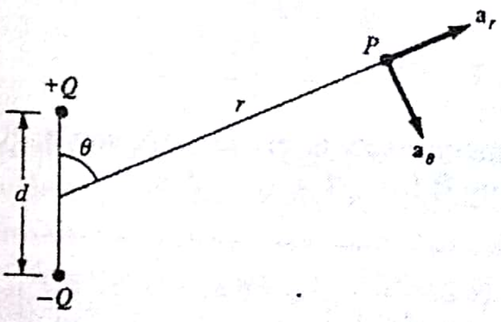
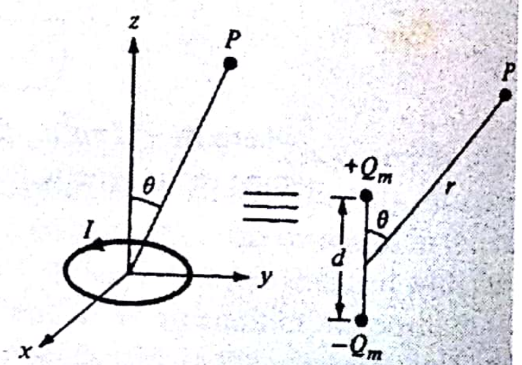
$$\boxed{\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta)} \tag{8.22}$$

It is interesting to compare eqs. (8.21) and (8.22) with similar expressions in eqs. (4.80) and (4.82) for electrical potential V and electric field intensity $\mathbf{E}$ due to an electric dipole. This comparison is done in Table 8.2, in which we notice the striking similarities between $\mathbf{B}$ in the far field due to a small current loop and $\mathbf{E}$ in the far field due to an electric dipole. It is therefore reasonable to regard a small current loop as a magnetic dipole. The $\mathbf{B}$ lines due to a magnetic dipole are similar to the $\mathbf{E}$ lines due to an electric dipole. Figure 8.7(a) illustrates the $\mathbf{B}$ lines around the magnetic dipole $\mathbf{m} = I\mathbf{S}$.

A short permanent magnetic bar, shown in Figure 8.7(b), may also be regarded as a magnetic dipole. Observe that the $\mathbf{B}$ lines due to the bar are similar to those due to a small current loop in Figure 8.7(a).

Consider the bar magnet of Figure 8.8. If Q_m is an isolated magnetic charge (pole strength) and ℓ is the length of the bar, the bar has a dipole moment $Q_m\ell$. (Notice that Q_m

Table 8.2 Comparison between Electric and Magnetic Monopoles and Dipoles

Electric	Magnetic
$V = \dfrac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \dfrac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$  Monopole (point charge)	Does not exist  Monopole (point charge)
$V = \dfrac{Q \cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \dfrac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$  Dipole (two-point charge)	$\mathbf{A} = \dfrac{\mu_o m \sin\theta\, \mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \dfrac{\mu_o m}{4\pi r^3}(2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$  Dipole (small current loop or bar magnet)

does exist; however, it does not exist without an associated $-Q_m$. See Table 8.2.) When the bar is in a uniform magnetic field $\mathbf{B}$, it experiences a torque

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m \boldsymbol{\ell} \times \mathbf{B} \tag{8.23}$$

where $\boldsymbol{\ell}$ points south to north. The torque tends to align the bar with the external magnetic field. The force acting on the magnetic charge is given by

$$\mathbf{F} = Q_m \mathbf{B} \tag{8.24}$$

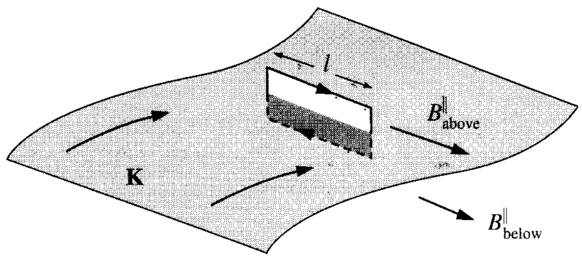


Figure 5.50

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}, \tag{5.75}$$

for $\nabla \cdot \mathbf{A} = 0$ guarantees[15] that the *normal* component is continuous, and $\nabla \times \mathbf{A} = \mathbf{B}$, in the form

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

means that the tangential components are continuous (the flux through an amperian loop of vanishing thickness is zero). But the *derivative* of $\mathbf{A}$ inherits the discontinuity of $\mathbf{B}$:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}. \tag{5.76}$$

Problem 5.31

(a) Check Eq. 5.74 for the configuration in Ex. 5.9.

(b) Check Eqs. 5.75 and 5.76 for the configuration in Ex. 5.11.

Problem 5.32 Prove Eq. 5.76, using Eqs. 5.61, 5.74, and 5.75. [*Suggestion:* I'd set up Cartesian coordinates at the surface, with z perpendicular to the surface and x parallel to the current.]

5.4.3 Multipole Expansion of the Vector Potential

If you want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. Remember: the idea of a multipole expansion is to write the potential in the form of a power series in $1/r$, where r is the distance to the point in question (Fig. 5.51); if r is sufficiently large, the series will be

[15]Note that Eqs. 5.75 and 5.76 presuppose that $\mathbf{A}$ is divergenceless.

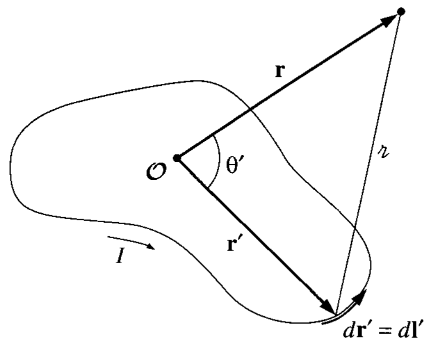


Figure 5.51

dominated by the lowest nonvanishing contribution, and the higher terms can be ignored. As we found in Sect. 3.4.1 (Eq. 3.94),

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta'). \tag{5.77}$$

Accordingly, the vector potential of a current loop can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\, d\mathbf{l}', \tag{5.78}$$

or, more explicitly:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\bigg[&\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\mathbf{l}' \\ &+ \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\mathbf{l}' + \cdots\bigg].\end{aligned} \tag{5.79}$$

As in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on.

Now, it happens that the *magnetic monopole term is always zero*, for the integral is just the total vector displacement around a closed loop:

$$\oint d\mathbf{l}' = 0. \tag{5.80}$$

This reflects the fact that there are (apparently) no magnetic monopoles in nature (an assumption contained in Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, on which the entire theory of vector potential is predicated).

In the absence of any monopole contribution, the dominant term is the dipole (except in the rare case where it, too, vanishes):

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'. \tag{5.81}$$

This integral can be rewritten in a more illuminating way if we invoke Eq. 1.108, with $\mathbf{c} = \hat{\mathbf{r}}$:

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'. \tag{5.82}$$

Then

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2},} \tag{5.83}$$

where $\mathbf{m}$ is the **magnetic dipole moment**:

$$\boxed{\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.} \tag{5.84}$$

Here $\mathbf{a}$ is the "vector area" of the loop (Prob. 1.61); if the loop is *flat*, $\mathbf{a}$ is the ordinary area enclosed, with the direction assigned by the usual right hand rule (fingers in the direction of the current).

Example 5.13

Find the magnetic dipole moment of the "bookend-shaped" loop shown in Fig. 5.52. All sides have length w, and it carries a current I.

Solution: This wire could be considered the superposition of two plane square loops (Fig. 5.53). The "extra" sides (AB) cancel when the two are put together, since the currents flow in opposite directions. The net magnetic dipole moment is

$$\mathbf{m} = Iw^2 \, \hat{\mathbf{y}} + Iw^2 \, \hat{\mathbf{z}};$$

its magnitude is $\sqrt{2} I w^2$, and it points along the 45° line $z = y$.

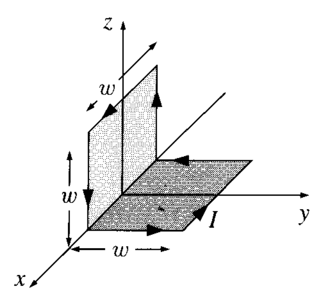


Figure 5.52

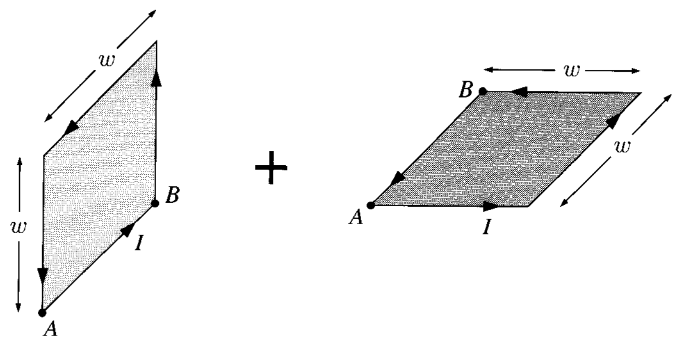


Figure 5.53

It is clear from Eq. 5.84 that the magnetic dipole moment is independent of the choice of origin. You may remember that the *electric* dipole moment is independent of the origin only when the total charge vanishes (Sect. 3.4.3). Since the *magnetic* monopole moment is *always* zero, it is not really surprising that the magnetic dipole moment is always independent of origin.

Although the dipole term *dominates* the multipole expansion (unless $\mathbf{m} = 0$), and thus offers a good approximation to the true potential, it is not ordinarily the *exact* potential; there will be quadrupole, octopole, and higher contributions. You might ask, is it possible to devise a current distribution whose potential is "pure" dipole—for which Eq. 5.83 is *exact*? Well, yes and no: like the electrical analog, it can be done, but the model is a bit contrived. To begin with, you must take an *infinitesimally small* loop at the origin, but then, in order to keep the dipole moment finite, you have to crank the current up to infinity, with the product $m = Ia$ held fixed. In practice, the dipole potential is a suitable approximation whenever the distance r greatly exceeds the size of the loop.

The magnetic *field* of a (pure) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z-direction (Fig. 5.54). According to Eq. 5.83, the potential at point

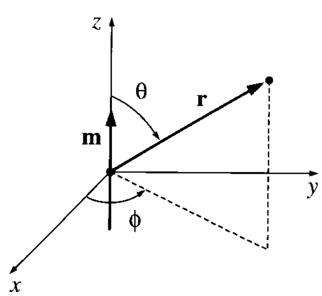
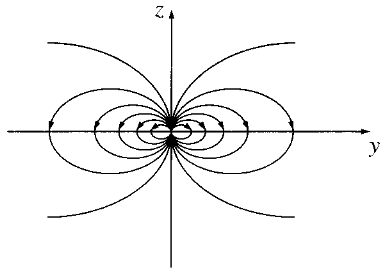
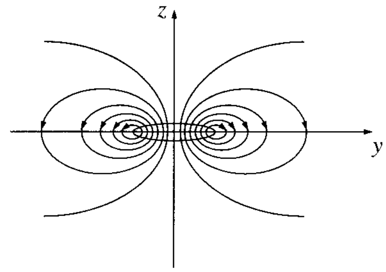


Figure 5.54



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Figure 5.55

(r, θ, ϕ) is

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\boldsymbol{\phi}}, \tag{5.85}$$

and hence

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{5.86}$$

Surprisingly, this is *identical* in structure to the field of an *electric* dipole (Eq. 3.103)! (Up close, however, the field of a *physical* magnetic dipole—a small current loop—looks quite different from the field of a physical electric dipole—plus and minus charges a short distance apart. Compare Fig. 5.55 with Fig. 3.37.)

• **Problem 5.33** Show that the magnetic field of a dipole can be written in coordinate-free form:

$$\boxed{\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}].} \tag{5.87}$$

Problem 5.34 A circular loop of wire, with radius R, lies in the xy plane, centered at the origin, and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?

(b) What is the (approximate) magnetic field at points far from the origin?

(c) Show that, for points on the z axis, your answer is consistent with the *exact* field (Ex. 5.6), when $z \gg R$.

Problem 5.35 A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment.

Problem 5.36 Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points $r > R$ the potential is that of a perfect dipole.

Problem 5.37 Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.