down by the field, so the *change* is *still* opposite to **B**.) Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But in the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and these increments are all *antiparallel* to the field. This is the mechanism responsible for **diamagnetism**. It is a universal phenomenon, affecting all atoms. However, it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with *even* numbers of electrons, where paramagnetism is usually absent.

In deriving Eq. 6.8 I assumed that the orbit remains circular, with its original radius R. I cannot offer a justification for this at the present stage. If the atom is stationary while the field is turned on, then my assumption can be proved—this is not magneto*statics*, however, and the details will have to await Chapter 7 (see Prob. 7.49). If the atom is moved into the field, the situation is enormously more complicated. But never mind—I'm only trying to give you a qualitative account of diamagnetism. Assume, if you prefer, that the velocity remains the same while the *radius* changes—the formula (6.8) is altered (by a factor of 2), but the *conclusion* is unaffected. The truth is that this classical model is fundamentally flawed (diamagnetism is really a *quantum* phenomenon), so there's not much point in refining the details.[3] What *is* important is the *empirical* fact that in diamagnetic materials the induced dipole moments point *opposite* to the magnetic field.

6.1.4 Magnetization

In the presence of a magnetic field, matter becomes *magnetized*; that is, upon microscopic examination it will be found to contain many tiny dipoles, with a net alignment along some direction. We have discussed two mechanisms that account for this magnetic polarization: (1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field) and (2) diamagnetism (the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field). Whatever the *cause*, we describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume}. \tag{6.9}$$

M is called the **magnetization**; it plays a role analogous to the polarization **P** in electrostatics. In the following section, we will not worry about how the magnetization *got* there—it could be paramagnetism, diamagnetism, or even ferromagnetism—we shall take **M** as *given*, and calculate the field this magnetization itself produces.

Incidentally, it may have surprised you to learn that materials other than the famous ferromagnetic trio (iron, nickel, and cobalt) are affected by a magnetic field *at all*. You cannot, of course, pick up a piece of wood or aluminum with a magnet. The reason is that diamagnetism and paramagnetism are extremely weak: It takes a delicate experiment and a powerful magnet to detect them at all. If you were to suspend a piece of paramagnetic

[3] S. L. O'Dell and R. K. P. Zia, *Am. J. Phys.* **54**, 32, (1986); R. Peierls, *Surprises in Theoretical Physics*, Section 4.3 (Princeton, N.J.: Princeton University Press, 1979); R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. 2, Sec. 34–36 (New York: Addison-Wesley, 1966).

material above a solenoid, as in Fig. 6.3, the induced magnetization would be upward, and hence the force downward. By contrast, the magnetization of a diamagnetic object would be downward and the force upward. In general, when a sample is placed in a region of nonuniform field, the *paramagnet is attracted into the field*, whereas the *diamagnet is repelled away*. But the actual forces are pitifully weak—in a typical experimental arrangement the force on a comparable sample of iron would be 10^4 or 10^5 times as great. That's why it was reasonable for us to calculate the field inside a piece of copper wire, say, in Chapter 5, without worrying about the effects of magnetization.

Problem 6.6 Of the following materials, which would you expect to be paramagnetic and which diamagnetic? Aluminum, copper, copper chloride ($CuCl_2$), carbon, lead, nitrogen (N_2), salt (NaCl), sodium, sulfur, water. (Actually, copper is slightly *dia*magnetic; otherwise they're all what you'd expect.)

6.2 The Field of a Magnetized Object

6.2.1 Bound Currents

Suppose we have a piece of magnetized material; the magnetic dipole moment per unit volume, **M**, is given. What field does this object produce? Well, the vector potential of a single dipole **m** is given by Eq. 5.83:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}. \tag{6.10}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}\,d\tau'$, so the total vector potential is (Fig. 6.11)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'. \tag{6.11}$$

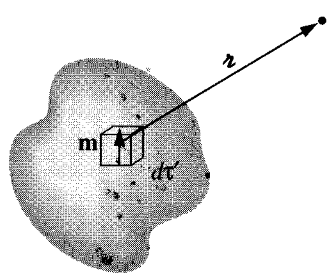

Figure 6.11

That *does* it, in principle. But as in the electrical case (Sect. 4.2.1), the integral can be cast in a more illuminating form by exploiting the identity

$$\nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

With this,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla' \frac{1}{\mathscr{r}} \right) \right] d\tau'.$$

Integrating by parts, using product rule 7, gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')]\, d\tau' - \int \nabla' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}} \right] d\tau' \right\}.$$

Problem 1.60(b) invites us to express the latter as a surface integral,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')]\, d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']. \tag{6.12}$$

The first term looks just like the potential of a *volume* current,

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M},} \tag{6.13}$$

while the second looks like the potential of a surface current,

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},} \tag{6.14}$$

where $\hat{\mathbf{n}}$ is the normal unit vector. With these definitions,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\, da'. \tag{6.15}$$

What this means is that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\mathbf{J}_b = \nabla \times \mathbf{M}$ throughout the material. plus a surface current $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$, on the boundary. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. 6.11, we first determine these **bound currents**, and then find the field *they* produce, in the same way we would calculate the field of any other volume and surface currents. Notice the striking parallel with the electrical case: there the field of a polarized object was the same as that of a bound volume charge $\rho_b = -\nabla \cdot \mathbf{P}$ plus a bound surface charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.

Example 6.1

Find the magnetic field of a uniformly magnetized sphere.

Solution: Choosing the z axis along the direction of $\mathbf{M}$ (Fig. 6.12), we have

$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\, \hat{\boldsymbol{\phi}}.$$

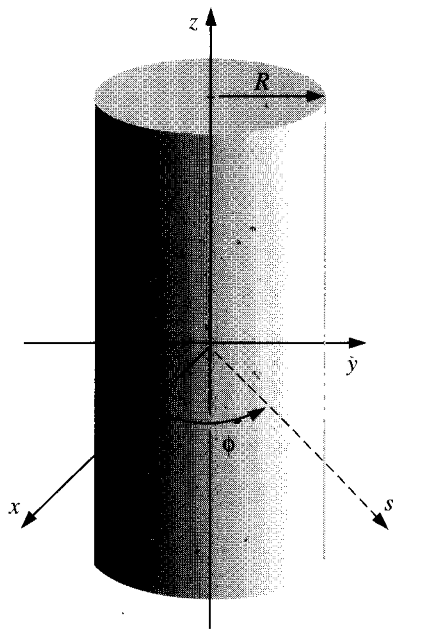

Figure 6.13

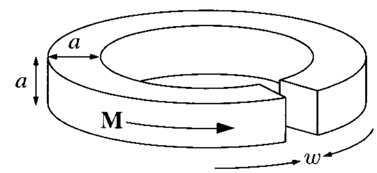

Figure 6.14

Problem 6.10 An iron rod of length L and square cross section (side a), is given a uniform longitudinal magnetization $\mathbf{M}$, and then bent around into a circle with a narrow gap (width w), as shown in Fig. 6.14. Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$. [*Hint:* treat it as the superposition of a *complete* torus plus a square loop with reversed current.]

6.2.2 Physical Interpretation of Bound Currents

In the last section we found that the field of a magnetized object is identical to the field that would be produced by a certain distribution of "bound" currents, $\mathbf{J}_b$ and $\mathbf{K}_b$. I want to show you how these bound currents arise physically. This will be a *heuristic* argument—the *rigorous* derivation has already been given. Figure 6.15 depicts a thin slab of uniformly magnetized material, with the dipoles represented by tiny current loops. Notice that all the "internal" currents cancel: every time there is one going to the right, a contiguous one is going to the left. However, at the edge there is *no adjacent loop to do the canceling*. The whole thing, then, is equivalent to a single ribbon of current I flowing around the boundary (Fig. 6.16).

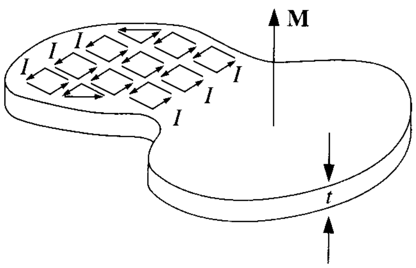

Figure 6.15

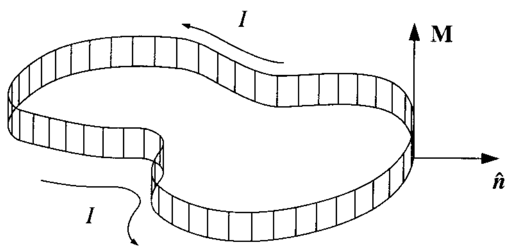

Figure 6.16

What *is* this current, in terms of **M**? Say that each of the tiny loops has area a and thickness t (Fig. 6.17). In terms of the magnetization M, its dipole moment is $m = Mat$. In terms of the circulating current I, however, $m = Ia$. Therefore $I = Mt$, so the surface current is $K_b = I/t = M$. Using the outward-drawn unit vector $\hat{\mathbf{n}}$ (Fig. 6.16), the direction of $\mathbf{K}_b$ is conveniently indicated by the cross product:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}.$$

(This expression also records the fact that there is *no* current on the top or bottom surface of the slab; here **M** is parallel to $\hat{\mathbf{n}}$, so the cross product vanishes.)

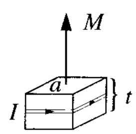

Figure 6.17

This bound surface current is exactly what we obtained in Sect. 6.2.1. It is a peculiar *kind* of current, in the sense that no single charge makes the whole trip—on the contrary, each charge moves only in a tiny little loop within a single atom. Nevertheless, the net effect is a macroscopic current flowing over the surface of the magnetized object. We call it a "bound" current to remind ourselves that every charge is attached to a particular atom, but it's a perfectly genuine current, and it produces a magnetic field in the same way any other current does.

When the magnetization is *non*uniform, the internal currents no longer cancel. Figure 6.18a shows two adjacent chunks of magnetized material, with a larger arrow on the one to the right suggesting greater magnetization at that point. On the surface where they join there is a net current in the x-direction, given by

$$I_x = [M_z(y + dy) - M_z(y)]\, dz = \frac{\partial M_z}{\partial y}\, dy\, dz.$$

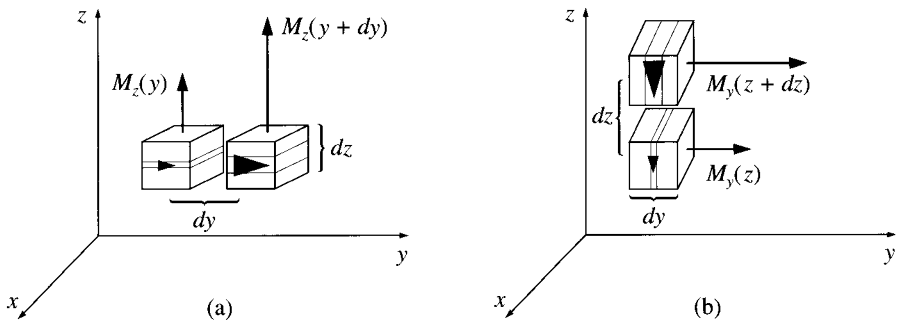

Figure 6.18

The corresponding volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}.$$

By the same token, a nonuniform magnetization in the y-direction would contribute an amount $-\partial M_y/\partial z$ (Fig. 6.18b), so

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$

In general, then,

$$\mathbf{J}_b = \nabla \times \mathbf{M},$$

consistent, again, with the result of Sect. 6.2.1. Incidentally, like any other steady current. $\mathbf{J}_b$ should obey the conservation law 5.31:

$$\nabla \cdot \mathbf{J}_b = 0.$$

Does it? *Yes*, for the divergence of a curl is *always* zero.

6.2.3 The Magnetic Field Inside Matter

Like the electric field, the actual *microscopic* magnetic field inside matter fluctuates wildly from point to point and instant to instant. When we speak of "the" magnetic field in matter. we mean the *macroscopic* field: the average over regions large enough to contain many atoms. (The magnetization $\mathbf{M}$ is "smoothed out" in the same sense.) It is this macroscopic field one obtains when the methods of Sect. 6.2.1 are applied to points inside magnetized material, as you can prove for yourself in the following problem.

Problem 6.11 In Sect, 6.2.1, we began with the potential of a *perfect* dipole (Eq. 6.10). whereas in *fact* we are dealing with *physical* dipoles. Show, by the method of Sect. 4.2.3, that we nonetheless get the correct macroscopic field.

In the presence of materials these are sometimes more useful than the corresponding boundary conditions on **B** (Eqs. 5.72 and 5.73):

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0, \tag{6.26}$$

and

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}). \tag{6.27}$$

You might want to check them, for Ex. 6.2 or Prob. 6.14.

Problem 6.14 For the bar magnet of Prob. 6.9, make careful sketches of **M**, **B**, and **H**, assuming L is about $2a$. Compare Prob. 4.17.

Problem 6.15 If $\mathbf{J}_f = 0$ everywhere, the curl of **H** vanishes (Eq. 6.19), and we can express **H** as the gradient of a scalar potential W:

$$\mathbf{H} = -\nabla W.$$

According to Eq. 6.23, then,

$$\nabla^2 W = (\nabla \cdot \mathbf{M}),$$

so W obeys Poisson's equation, with $\nabla \cdot \mathbf{M}$ as the "source." This opens up all the machinery of Chapter 3. As an example, find the field inside a uniformly magnetized sphere (Ex. 6.1) by separation of variables. [*Hint:* $\nabla \cdot \mathbf{M} = 0$ everywhere except at the surface ($r = R$), so W satisfies Laplace's equation in the regions $r < R$ and $r > R$; use Eq. 3.65, and from Eq. 6.24 figure out the appropriate boundary condition on W.]

6.4 Linear and Nonlinear Media

6.4.1 Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when **B** is removed, **M** disappears. In fact, for most substances the magnetization is *proportional* to the field, provided the field is not too strong. For notational consistency with the electrical case (Eq. 4.30), I *should* express the proportionality thus:

$$\mathbf{M} = \frac{1}{\mu_0}\chi_m \mathbf{B} \quad \text{(incorrect!)}. \tag{6.28}$$

But custom dictates that it be written in terms of **H**, instead of **B**:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}.} \tag{6.29}$$

The constant of proportionality χ_m is called the **magnetic susceptibility**; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5} (see Table 6.1).

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen (−200° C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics,* 67th ed. (Boca Raton: CRC Press, Inc., 1986).

Materials that obey Eq. 6.29 are called **linear media.** In view of Eq. 6.18,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}, \tag{6.30}$$

for linear media. Thus **B** is *also* proportional to **H**:[5]

$$\mathbf{B} = \mu\mathbf{H}, \tag{6.31}$$

where

$$\mu \equiv \mu_0(1 + \chi_m). \tag{6.32}$$

μ is called the **permeability** of the material.[6] In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the **permeability of free space**.

Example 6.3

An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

Solution: Since **B** is due in part to bound currents (which we don't yet know), we cannot compute it directly. However, this is one of those symmetrical cases in which we can get **H** from the free current alone, using Ampère's law in the form of Eq. 6.20:

$$\mathbf{H} = nI\,\hat{\mathbf{z}}$$

[5]Physically, therefore, Eq. 6.28 would say exactly the same as Eq. 6.29, only the constant χ_m would have a different value. Equation 6.29 is a little more convenient, because experimentalists find it handier to work with **H** than **B**.

[6]If you factor out μ_0, what's left is called the **relative permeability**: $\mu_r \equiv 1 + \chi_m = \mu/\mu_0$. By the way, formulas for **H** in terms of **B** (Eq. 6.31, in the case of linear media) are called **constitutive relations**, just like those for **D** in terms of **E**.

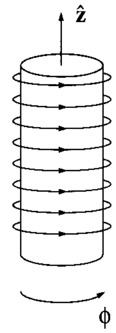

Figure 6.22

(Fig. 6.22). According to Eq. 6.31, then,

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}.$$

If the medium is paramagnetic, the field is slightly enhanced; if it's diamagnetic, the field is somewhat reduced. This reflects the fact that the bound surface current

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m(\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m nI\,\hat{\boldsymbol{\phi}}$$

is in the same direction as I, in the former case ($\chi_m > 0$), and opposite in the latter ($\chi_m < 0$).

You might suppose that linear media avoid the defect in the parallel between **B** and **H**: since **M** and **H** are now proportional to **B**, does it not follow that their divergence, like **B**'s, must always vanish? Unfortunately, it does *not*; at the *boundary* between two materials of different permeability the divergence of **M** can actually be infinite. For instance, at the end of a cylinder of linear paramagnetic material, **M** is zero on one side but not on the other. For the "Gaussian pillbox" shown in Fig. 6.23, $\oint \mathbf{M} \cdot d\mathbf{a} \neq 0$, and hence, by the divergence theorem, $\nabla \cdot \mathbf{M}$ cannot vanish everywhere within.

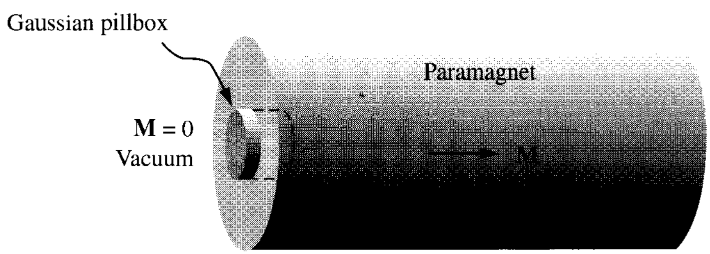

Figure 6.23

Incidentally, the volume bound current density in a homogeneous linear material is proportional to the *free* current density:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f. \tag{6.33}$$

In particular, unless free current actually flows *through* the material, all bound current will be at the surface.

Problem 6.16 A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface (Fig. 6.24). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

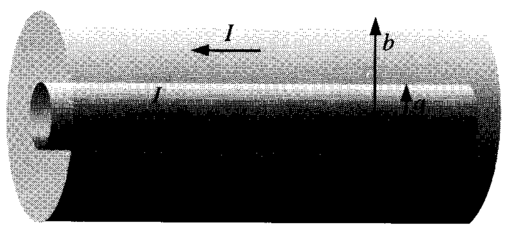

Figure 6.24

Problem 6.17 A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the *net* bound current flowing down the wire?

! **Problem 6.18** A sphere of linear magnetic material is placed in an otherwise uniform magnetic field $\mathbf{B}_0$. Find the new field inside the sphere. [*Hint:* See Prob. 6.15 or Prob. 4.23.]

Problem 6.19 On the basis of the naïve model presented in Sect. 6.1.3, estimate the magnetic susceptibility of a diamagnetic metal such as copper. Compare your answer with the empirical value in Table 6.1, and comment on any discrepancy.
