E-Materials (II)

Course Name: B.Sc.(H) Mathematics (II Year, IV Semester)

Paper Name: Partial Differential Equations

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Canonical Form (Continue**)**

(A) Hyperbolic Type

If $B^2 - 4AC > 0$, then integration of equations (4.2.5) and (4.2.6) yield two real and distinct families of characteristics. Equation (4.1.11) reduces to

$$u_{\xi\eta} = H_1, \tag{4.2.7}$$

where $H_1 = H^*/B^*$. It can be easily shown that $B^* \neq 0$. This form is called *the first canonical form of the hyperbolic equation.*

Now if new independent variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta, \tag{4.2.8}$$

are introduced, then equation (4.2.7) is transformed into

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta) \tag{4.2.9}$$

This form is called the *second canonical form of the hyperbolic equation.*

(B) Parabolic Type

In this case, we have $B^2 - 4AC = 0$, and equations (4.2.5) and (4.2.6) coincide. Thus, there exists one real family of characteristics, and we obtain only a single integral $\xi =$ constant (or $\eta =$ constant).

Since $B^2 = 4AC$ and $A^* = 0$, we find that

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = \left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)^2 = 0.$$

From this it follows that

$$
\begin{aligned}
A^* &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2\left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)\left(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y\right) = 0,
\end{aligned}
$$

for arbitrary values of $\eta(x,y)$ which is functionally independent of $\xi(x,y)$; for instance, if $\eta = y$, the Jacobian does not vanish in the domain of parabolicity.

Division of equation (4.1.11) by C^* yields

$$
u_{\eta\eta} = H_3\left(\xi, \eta, u, u_\xi, u_\eta\right), \qquad C^* \neq 0. \tag{4.2.10}
$$

This is called the *canonical form of the parabolic equation.*

Equation (4.1.11) may also assume the form

$$
u_{\xi\xi} = H_3^*\left(\xi, \eta, u, u_\xi, u_\eta\right), \tag{4.2.11}
$$

if we choose $\eta =$ constant as the integral of equation (4.2.5).

(C) Elliptic Type

For an equation of elliptic type, we have $B^2 - 4AC < 0$. Consequently, the quadratic equation (4.2.4) has no real solutions, but it has two complex conjugate solutions which are continuous complex-valued functions of the real variables x and y. Thus, in this case, there are no real characteristic curves. However, if the coefficients A, B, and C are analytic functions of x and y, then one can consider equation (4.2.4) for complex x and y. A function of two real variables x and y is said to be analytic in a certain domain if in some neighborhood of every point (x_0, y_0) of this domain, the function can be represented as a Taylor series in the variables $(x - x_0)$ and $(y - y_0)$.

Since ξ and η are complex, we introduce new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \qquad \beta = \frac{1}{2i}(\xi - \eta), \tag{4.2.12}$$

so that

$$\xi = \alpha + i\beta, \qquad \eta = \alpha - i\beta. \tag{4.2.13}$$

First, we transform equations (4.1.10). We then have

$$A^{**}(\alpha,\beta)\, u_{\alpha\alpha} + B^{**}(\alpha,\beta)\, u_{\alpha\beta} + C^{**}(\alpha,\beta)\, u_{\beta\beta} = H_4(\alpha,\beta,u,u_\alpha,u_\beta), \tag{4.2.14}$$

in which the coefficients assume the same form as the coefficients in equation (4.1.11). With the use of (4.2.13), the equations $A^* = C^* = 0$ become

$$\left(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2\right) - \left(A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2\right) + i\left[2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y\right] = 0,$$

$$\left(A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2\right) - \left(A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2\right) - i\left[2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y\right] = 0,$$

or,

$$(A^{**} - C^{**}) + iB^{**} = 0, \qquad (A^{**} - C^{**}) - iB^{**} = 0.$$

These equations are satisfied if and only if

$$A^{**} = C^{**} \quad \text{and} \quad B^{**} = 0.$$

Hence, equation (4.2.14) transforms into the form

$$A^{**}u_{\alpha\alpha} + A^{**}u_{\beta\beta} = H_4(\alpha,\beta,u,u_\alpha,u_\beta).$$

Dividing through by A^{**}, we obtain

$$u_{\alpha\alpha} + u_{\beta\beta} = H_5(\alpha,\beta,u,u_\alpha,u_\beta), \tag{4.2.15}$$

where $H_5 = (H_4/A^{**})$. This is called the *canonical form of the elliptic equation.*

) *Example 4.2.1.* Consider the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

Here

$$A = y^2, \quad B = 0, \quad C = -x^2.$$

Thus,

$$B^2 - 4AC = 4x^2y^2 > 0.$$

The equation is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$. From the characteristic equations (4.2.5) and (4.2.6), we have

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

After integration of these equations, we obtain

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1, \qquad \frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2.$$

The first of these curves is a family of hyperbolas

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1,$$

and the second is a family of circles

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2.$$

To transform the given equation to canonical form, we consider

$$\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2, \qquad \eta = \frac{1}{2}y^2 + \frac{1}{2}x^2.$$

From the relations (4.1.6), we have

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -x u_\xi + x u_\eta, \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = y u_\xi + y u_\eta, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
&= x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta. \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} \\
&= y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta.
\end{aligned}$$

Thus, the given equation assumes the canonical form

$$u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)} u_\eta.$$

Example 4.2.2. Consider the partial differential equation

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0.$$

In this case, the discriminant is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$$

The equation is therefore parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = \frac{y}{x},$$

and hence, the characteristics are

$$\frac{y}{x} = c,$$

which is the equation of a family of straight lines.

Consider the transformation

$$\xi = \frac{y}{x}, \quad \eta = y,$$

where η is chosen arbitrarily. The given equation is then reduced to the canonical form

$$y^2 u_{\eta\eta} = 0.$$

Thus,

$$u_{\eta\eta} = 0 \quad \text{for} \quad y \neq 0.$$

Example 4.2.3. The equation

$$u_{xx} + x^2 u_{yy} = 0,$$

is elliptic everywhere except on the coordinate axis $x = 0$ because

$$B^2 - 4AC = -4x^2 < 0, \quad x \neq 0.$$

The characteristic equations are

$$\frac{dy}{dx} = ix, \qquad \frac{dy}{dx} = -ix.$$

Integration yields

$$2y - ix^2 = c_1, \qquad 2y + ix^2 = c_2.$$

Thus, if we write

$$\xi = 2y - ix^2, \qquad \eta = 2y + ix^2,$$

and hence,

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y, \qquad \beta = \frac{1}{2i}(\xi - \eta) = -x^2,$$

we obtain the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta.$$

It should be remarked here that a given partial differential equation may be of a different type in a different domain. Thus, for example, *Tricomi's equation*

$$u_{xx} + x u_{yy} = 0, \tag{4.2.16}$$

is elliptic for $x > 0$ and hyperbolic for $x < 0$, since $B^2 - 4AC = -4x$.