E-Materials V

Course Name: B.Sc. (H) Geology, B.Sc. (H) Electronics, B.A (H) Economics

(II Year, IV Semester)

Generic Elective: GE4: Numerical Methods

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Runge Kutta Method

Taylor series method of solving differential equations is very tedious as it requires finding higher order derivatives. Runge Kutta (RK) method does not require calculation of higher order derivatives and gives greater accuracy. RK is the generalization of the concept used in modified Euler's method. They are all based on the general form of the extrapolation equation

$$y_{i+1} = y_i + \text{slope} \times \text{interval size}$$

$= y_i + mh$, m is the slope that is the weighted average of slopes at various points in intervals h.

Suppose we have to estimate m using slopes at r points in the interval (x_i, x_{i+1}), then m can be written as

$$m = w_i m_i + w_2 m_2 + \dots + w_r m_r$$

where w_i, $i = 1$ to r are weight of the slope

$$m_1 = f(x_i, y_i)$$

$$m_2 = f(x_i + a_1 h, y_i + b_{11} m_1 h)$$

$$m_r = f(x_i + a_{r-1} h, y_i + b_{r-1}, m_1 h + \dots + b_{r-1, r-1}, m_{r-1} h).$$

First Order R.K. Method

We know that Euler's method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy'_0 [y' = f(x, y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h)$$

$$= y_0 + hy'_0 + \frac{h^2}{2} \times y''_0 + \dots$$

Therefore, Euler's method agrees with Taylor's series solution up to the term in h.

Second Order R. K. Method

We know that modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ in the previous equation, we get

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)] \quad [\text{Since}, f_0 = f(x_0, y_0)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' \qquad \text{...(1)}$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series, we get

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f(x_0, y_0) + h\left(\frac{df}{dx}\right)_0 + hf_0\left(\frac{df}{dy}\right)_0 + 0(h^2)\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{df}{dx}\right)_0 + f_0\left(\frac{df}{dy}\right)_0\right\} + 0(h^3)\right]$$

$$= y_0 + \frac{1}{2}[2hf_0 + h^2 f_0' + 0(h^3)]$$

$$= y_0 + hf_0 + \frac{h^2}{2}f_0' + 0(h^3)$$

$$= y_0 + hy_0' + \frac{h^2}{2}y_0'' + 0(h^3) \qquad \text{...(ii)} \ [\therefore y^r = f^{r-1}(x, y)]$$

Comparing (i) and (ii) it follows that modified Euler's method agrees with Taylor series solution upto terms in h^2.

Hence, it is RK method of 2nd order, the formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_0, y_0),\ k_2 = hf(x_0 + h, y_0 + k_1).$$

Example 1. Given $y' + y = 0$, find $y(0.1)$ and $y(0.2)$ using R.K. second order method for $y(0) = 1$.

Solution: By R. K. second order method $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$y' = -y \text{ therefore } f(x, y) = -y$$

$$y(0) = 1, x_0 = 0, y_0 = 1, h = 0.1$$

$$x_1 = 0.1, y(0.1) = ?$$

$$k_1 = hf(x_0, y_0) = (0.1) f(0, 1)$$

$$k_1 = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.1, 0.9)$$

$$= 0.1(-0.9) = -0.09$$

$$y_1 = 1 + \frac{1}{2}(-0.1 - 0.09) = 0.905$$

$$y(0.1) = 0.905$$

Similarly proceeding for $y(0.2)$ we get

$$y(0.2) = 0.819025.$$

Third Order R. K. Method

Given $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$

Calculating successively as in the previous way

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf(x_0 + h, y_0 + k')$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

which gives the required approximate values

$\Rightarrow$ $y_1 = y_0 + k$ (k is weighted average of k_1, k_2, k_3)

Fourth Order R. K. Method

It is the most common and referred to as Runge Kutta method only.

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Calculating successively as in the previous way

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate values

$$y_1 = y_0 + k \text{ (} k \text{ is weighted average of } k_1, k_2, k_3, k_4\text{).}$$

Example 2. Apply Runge Kutta fourth order method to find an approximate value of y when x = 0.3, given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution: Given $x_0 = 0, y_0 = 1, h = 0.3, f(x_0, y_0) = 1$

$$k_1 = hf(x_0, y_0) = 0.3(1) = 0.3$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= hf(0.15, 1.15) = 0.3(1.3) = 0.39$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= hf(0.15, 1.195) = 0.3(1.345) = 0.4035$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= hf(0.3, 1.4035) = 0.3(1.7035) = 0.51105$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.3 + 0.78 + 0.807 + 0.51105) = 0.399675$$

Hence, the required approximate value is 1.399675 [As $y_1 = y_0 + k$].

Example 3. Using Runge Kutta method of order 4, solve

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \text{ with } y(0) = 1 \text{ at } x = 0.5.$$

Solution: Given $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

$$x_0 = 0, y_0 = 1, h = 0.5$$

$$f(x_0, y_0) = \frac{1 - 0}{1 + 0} = 1$$

$$k_1 = hf(x_0, y_0) = 0.5(1) = 0.5$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}\right) = 0.5 f(0.25, 1.25)$$

$$= (0.5)\frac{(1.25)^2 - (0.25)^2}{(1.25)^2 + (0.25)^2} = (0.5)\frac{1.5625 - 0.0625}{1.5625 + 0.0625}$$

$$= 0.5(0.923) = 0.4615$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}\right) = 0.5 f(0.25, 1.23)$$

$$= (0.5)\frac{(1.23)^2 - 0.0625}{(1.23)^2 + 0.0625} = 0.5(0.9206) = 0.4603$$

$$k_4 = 0.5 f(x_0 + h, y_0 + k_3)$$

$$= 0.5 f(0.5, 1.4603) = (0.5)\frac{(1.4603)^2 - (0.5)^2}{(1.4603)^2 + (0.5)^2}$$

$$= 0.5(0.7901) = 0.39507$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.5 + 0.923 + 0.9206 + 0.39507) = 0.456445$$

Therefore, the required approximate value of y is 1. 456445.

Example 4. Apply Runge Kutta Method to find approximate value of y for x = 0.2 in steps of 0.1, if $\frac{dy}{dx} = x + y^2$ given that y = 1 when x = 0.

Solution: Given $f(x, y) = x + y^2$

$$x_0 = 0, y_0 = 1, h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= hf(0, 1) = 0.1(1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 1.05)$$

$$= 0.1(0.05 + (1.05)^2) = 0.11525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2}\right) = 0.1 f(0.05, 1.057625)$$

$$= 0.1(0.05 + (1.057625)^2) = 0.116857$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1)\ f(0.1, 1.116857) = 0.134737$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.2305 + 0.233714 + 0.134737) = 0.1165$$

$$y(0.1) = y_0 + h = 1.1165$$

$$x_1 = 0.1, y_1 = 1.1165, h = 0.1$$

$$k_1 = hf(x_1, y_1) = hf(0.1, 1.1165) = 0.13466$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.18383)$$

$$= 0.15514$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1\ f(0.15, 1.19407) = 0.1576$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1\ f(0.2, 1.2741) = 0.1823$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence, $y(0.2)$ is $y_1 + k = 1.2736$.

Heun's Method (Based on R-K Method)

Second order Runge-Kutta method is also known as Heun's method. Its derivation is the same as R-K second order method.

We know that second order R-K method has the form

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)h \qquad ...(1)$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11}k_1 h)$$

The weights a_1, a_2 and constants p_1, q_{11} are to be determined.

We also know that the principle of R.K approach is that these parameters are chosen such that the power series expansion of $y_{i+1} = y_i + (a_1k_1 + a_2k_2)h$. Here right side agrees with the Taylor series expansion of y_{i+1} in terms of y_i and $f(x_i, y_i)$.

Second order Taylor series expansion is

$$y_i + yh + \frac{y'' h^2}{2!}$$

Where,

$$y_i' = f(x_i, y_i) = f$$

$$y''_i = f(x_i, y_i) = f$$

$$y''_i = \frac{df}{dx} = f_x + ff_y$$

$\therefore$ Equation becomes $y_{i+1} = y_i + fh + (f_x + f_y f)\dfrac{h^2}{2}$ ...(2)

Now consider equation (1)

We need to expand k_2 as a power series function in terms of $f(x_i, y_i)$ as k_i is already a function of x_i and y_i

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_i h)$$
$$k_2 = f(x_i + p_1 h, f_x + q_{11} k_i h\, fy + 0\,(h^2))$$

Substituting in equation (1), we get

$$y_{i+1} = f_i + [q_1 f + a_2 p_1 h\, f_x + a_2 q_{11} h\, ff_y]h + 0\,(h^3)$$
$$= y_i + (a_1 + a_2)\, hf + (a_2 p_1 f_x + a_2 q_{11} ff_y)h^2 + 0(h^3)$$

equation (2) = equation (3)

This is possible only if

$$a_1 + a_2 = 1$$
$$a_2 p_1 = \frac{1}{2}$$
$$a_2 q_{11} = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns and the other three will be then determined from the other three equations.

(i) Heun's Method

Here, $a_2 = \dfrac{1}{2}$ ischosen, giving

$$p_1 = 1$$
$$q_1 = 1$$
$$q_{11} = 1$$

Giving $y_{i+1} = y_i + \left(\dfrac{1}{2}k_1 + \dfrac{1}{2}k_2\right)h$

where
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

Note: This is the same formula as of R. K. order 2.

(ii) Mid-point Method

Here $a_2 = 1$ giving $a_1 = 0$

$$p_1 = \frac{1}{2}$$
$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+2} = y_i + k_2 h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

(iii) Ralston's Method

Here $a_2 = \frac{2}{3}$

giving $a_1 = \frac{1}{3}$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in $$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Example 5. Given the equation $\frac{dy}{dx} = \frac{2y}{x}$ with $y(1) = 2$. Estimate $y(2)$ using Heun's method, Mid-point method and Ralston method with $h = 0.25$.

Solution: Using Heun's method

Iteration 1

$$x_0 = 1, y_0 = 2, h = 0.25$$

$$k_1 = 0.25 f(x_0, y_0) = 0.25 f(1, 2) = 1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.25 f(1.25, 3)$$

$$= (0.25)\frac{2(3)}{1.25} = 1.2$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$y_1 = 2 + \frac{1}{2}(1.2 + 1) = 3.1$$

$$y_1 = 3.1$$

Iteration 2

$$x_1 = 1.25, y_1 = 3.1$$
$$k_1 = hf(x_i, y_i) = 0.25\, f(1.25, 3.1)$$
$$= (0.25)\frac{2(3.1)}{1.25} = 1.24$$
$$k_2 = hf(x_i + h, y_1 + k_1) = 0.25\, f(1.5, 4.34)$$
$$= (0.25)\frac{2(4.34)}{1.5} = 1.44$$
$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$
$$= 3.1 + \frac{1}{2}(1.24 + 1.44) = 4.44$$
$$y_2 = 4.44$$

Proceeding like this we get the following table:

x	y
1	2
1.25	3.1
1.5	4.44
1.75	6.03
2	7.86

Using Mid-Point Method

Iteration 1

$$x_0 = 1, y_0 = 2, h = 0.25$$
$$k_1 = hf(x_0, y_0)$$
$$= 0.25\, f(1, 2) = (0.25)\frac{2(2)}{1} = 1$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= f(1.125, 2.5) = \frac{2(2.5)}{1.125} = 4.444$$
$$y_1 = y_0 + k_2 h$$
$$= 2 + 4.444(0.25)$$
$$y_1 = 3.11$$

Iteration 2

$$\begin{aligned} x_1 &= 1.25, y_1 = 3.11, h = 0.25 \\ k_1 &= hf(x_1, y_1) = 0.25\, f(1.25, 3.11) \\ &= (0.25)\frac{3.11(2)}{1.25} = 1.244 \end{aligned}$$

$$\begin{aligned} k_2 &= \left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\ &= f(1.375, 3.732) = \frac{2(3.732)}{1.375} = 5.428 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + k_2 h \\ &= 3.11 + 5.428\ (0.25) = 4.467 \end{aligned}$$

Iteration 3

$$\begin{aligned} x_2 &= 1.5\ y_2 = 4.467, h = 0.25 \\ k_1 &= hf(x_2, y_2) \\ &= 0.25 f(1.5, 4.467) = (0.25)\frac{2(4.467)}{1.5} = 1.489 \\ k_2 &= f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) \\ &= f(1.625, 5.2115) = \frac{2(5.2115)}{1.625} = 6.41415 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + k_2 h \\ &= 4.467 + (6.41415)(0.25) = 6.07 \end{aligned}$$

Proceeding like this we can compute the entire table

x	y
1	2
1.25	3.11
1.5	4.467
1.75	6.07
2	7.3996

Using Ralston's Method

Iteration 1

$$\begin{aligned} x_0 &= 1, y_0 = 2, h = 0.25 \\ k_1 &= hf(x_0, y_0) = 0.25\, f(1, 2) = 1 \\ k_2 &= f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}k_1\right) \\ &= f(1.1875, 2.75) = 4.632 \end{aligned}$$

$$y_1 = y_0 + \frac{h}{3}(k_1 + 2k_2)$$

$$= 2 + \frac{0.25}{3}(1 + 2(4.632)) = 2.855$$

Iteration 2

$$y_1 = 2.855,\ x_1 = 1.25$$

$$k_1 = hf(x_1, y_1) = 0.25\, f(1.25,\ 2.855) = 1.142$$

$$k_2 = f\left(x_1 + \frac{3}{4}h,\ y_1 + \frac{3}{4}k_1\right)$$

$$= \mathrm{f}(1.4375,\ 3.7115) = 5.1638$$

$$y_2 = y_1 + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h = 3.8108$$

Iteration 3

$$x_2 = 1.5,\ y_2 = 3.8108$$

$$k_1 = hf(x_2, y_2) = 0.25\, f(1.5,\ 3.8108) = 1.273$$

$$k_2 = f\left(x_2 + \frac{3}{4}h,\ y_2 + \frac{3}{4}k_1\right)$$

$$= f(1.6875,\ 4.7655) = 5.648$$

$$y_3 = y_2 + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h = 4.858$$

Similarly we can evaluate y_4 and tabulate the given results.

x	y
1	2
1.25	2.855
1.5	3.8108
1.75	4.858
2	6.0893

Example 6. Solve the following equation using R.K. 4th order and Mid-point method $\frac{dy}{dx} = yx^3 - 1.5y$ given $y(0) = 1$, $h = 0.5$ for the interval (0, 2).

Solution: Iteration 1

We have $x_0 = 0$, $y_0 = 1$

From R-K 4th order

$$f(x_0, y_0) = f(0,\ 1)$$

$$k_1 = hf(0,\ 1) = -0.75$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.5\ \mathrm{f}(0.25,\ 0.625) = -0.464$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.25, 0.768) = -0.57$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= \text{hf}(0.5, 0.43) = -0.2956$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.5189$$

$$y_1 = y_0 + k = 0.48106$$

Iteration 2

$$x_1 = 0.5, y_1 = 0.48106, f(x_1, y_1) = -0.6614575$$
$$k_1 = hf(x_1, y_1) = -0.3307$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.5\ \text{f}(0.75, 0.31571) = -0.1702$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.5\, f(0.75, 0.39596) = -0.213$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= 0.5\ \text{f}(1, 0.26806) = -0.067$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.194$$

$$y_2 = y_1 + k = 0.28706$$

Iteration 3

$$x_2 = 1, y_2 = 0.28706, f(x_2, y_2) = -0.14353$$
$$k_1 = hf(x_2, y_2) = -0.071765$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= 0.5\, f(1.25, 0.2512) = 0.0569$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= hf(1.25, 0.3155) = 0.0715$$
$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.5\, f(1.5, 0.35856) = 0.33615$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.08686$$

$$y_3 = y_2 + k = 0.37392$$

Iteration 4

$$x_3 = 1.5, y_3 = 0.37392$$
$$f(1.5, 0.37392) = 0.7011$$
$$k_1 = hf(1.5, 0.37392) = 0.5(0.7011) = 0.35055$$
$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right)$$
$$= 0.5\ (f(1.75, 0.5492)) = 1.0598$$
$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right)$$
$$= 0.5\ f(1.75, 0.90382) = 1.7441$$
$$k_4 = hf(x_3 + h, y_3 + k_3)$$
$$= 0.5\ f(2, 2.118) = 6.8835$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.14$$
$$y_4 = y_3 + k = 2.51395$$

Mid-Point Method

Iteration 1

$$x_0 = 0, y_0 = 1, h = 0.5, f(0, 1) = -1.5$$
$$k_1 = hf(x_0, y_0) = 0.5\ f(0, 1) = -0.75$$
$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) = f(0.25, 0.625) = -0.927$$
$$= y_1 = y_0 + k_2 h = 0.5365$$

Iteration 2

$$x_1 = 0.5, y_1 = 0.5365$$
$$k_1 = hf(x_1, y_1)$$
$$= 0.5\ f(0.5, 0.5365) = -0.3688$$
$$k_2 = f(0.75, 0.352) = -0.3795$$
$$y_2 = y_1 + k_2 h = 0.3468$$

Iteration 3

$$x_2 = 1, y_2 = 0.3468$$
$$k_1 = hf(1, 0.3468) = 0.5(-0.1734) = -0.0867$$
$$k_2 = f(1.25, 0.30345) = 0.1375$$
$$y_3 = y_2 + k_2 h = 0.41555$$

Iteration 4

$$x_3 = 1.5, y_3 = 0.41555$$
$$k_1 = hf(x_3, y_3)$$
$$= 0.5\ f(1.5, 0.41555) = 0.5(0.7792) = 0.3896$$

$$k_2 = f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = f(1.75, 0.6104) = 2.356$$

$$y_4 = y_3 + k_2 h = 1.594$$

The results of both methods can be tabulated

R-K 4th order		**Mid-Point Method**	
x	y	x	y
0	1	0	1
0.5	0.48106	0.5	0.5365
1	0.28706	1	0.3468
1.5	0.37392	1.5	0.41555
2	2.51392	2	1.594