# E-Materials IV

**Course Name:** B.Sc. (H) Geology, B.Sc. (H) Electronics, B.A (H) Economics

(II Year, IV Semester)

**Generic Elective:** GE4: Numerical Methods

**Faculty Name:** Mahendra Ram

# Chapter 8

## Ordinary Differential Equations

### Introduction

A differential equation is an equation, where the unknown is a function and both the function and its derivative may appear in the equation. Differential equations are essential for a mathematical description of nature. They are the core of many physical theories. For example, Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism and Einstein's equation for the general theory of gravitation. When the function involves only one independent variable then the equation is called an Ordinary differential equation or ODE.
**For example:** In Radioactive decay, the amount '*u*' of a radioactive material changes in time as follows,

$$\frac{du}{dt}(t) = -ku(t), k > 0$$

where k is a positive constant representing radioactive properties of the material. Differential equations are also classified according to their order. In radioactive decay, the equation is first order ordinary differential equation because the highest derivative is a first derivative. A second order equation would include a second derivative. For example, Newton's law: Mass times acceleration equals force, t = na, where n is the particle mass,

$$a = \frac{d^2x}{dt^2}(t) = t\left(t, x(t), \frac{dx}{dt}(t)\right)$$

where the unknown is $x(t)$– the position of the particle in space at the time $t$. As we see above, the force may depend on time, on the particle position in space, and on the particle velocity. This is a second order ordinary differential equation (ODE). Similarly, when dealing with nth order differential equations, $n$ conditions are required to obtain a unique solution.

**Euler's Method**

In mathematics and computational sciences, the Euler's method is the first order numerical procedure for solving ordinary differential equations with a given initial condition. It is the most basic and explicit method for numerical integration of ordinary differential equations. Leonard Euler was a famous Swiss mathematician. He contributed to all branches of mathematics, especially its application to the problems of physics. He invented a method to compute the numerical solution of a first order and first degree differential equation. The method is as follows: Consider a first order differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0 \qquad ...(1)$$

We divide the interval $(x_0, x_n)$ into $n$ sub intervals, each of width $h$ so that $x_n = x_0 + nh$ for $n = 1, 2, 3$. Now, we wish to find the value of $y$ at $x_n = x_0 + nh$. In the interval $(x_0, x_1)$ we approximate the curve $y(x)$ by the tangent line at $(x_0, y_0)$ whose slope is

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = f(x_0, y_0)$$

The equation of line through $(x_0, y_0)$ whose slope is $f(x_0, y_0)$, is given by –

$$y - y_0 = [f(x_0, y_0)] (x - x_0)$$

If the ordinate corresponding to $x_1$ meets this tangent line at $(x_1, y_1)$, then

$$y_1 - y_0 = (x_1 - x_0) f(x_0, y_0)$$

$$y_1 = y_0 + hf(x_0, y_0) \; (h = x_1 - x_0)$$

(because width of each interval is $h$)

Again in the interval $(x_1, x_2)$ and through the point $(x_1, y_1)$, we approximate the curve $y(x)$ by the tangent line at $(x_1, y_1)$ whose slope is $\left(\frac{dy}{dx}\right)(x_1, y_1) = f(x_1, y_1)$.

Then the equation of the tangent line is–

$$y - y_1 = [f(x_1, y_1)] (x - x_1)$$

If the ordinate corresponding to $x_2$ meet this tangent line at $(x_2, y_2)$, then

$$y_2 - y_1 = (x_2 - x_1)\, f(x_1, y_1)$$

$$y_2 = y_1 + hf(x_1, y_1) \qquad [\because\ h = x_2 - x_1]$$

Continuing like this '$n$' times we get

$$y_{n+1} = y_n + hf(x_n, y_n),\ \ n = 0, 1, 2, ...$$

Graphical representation of **Euler's method**

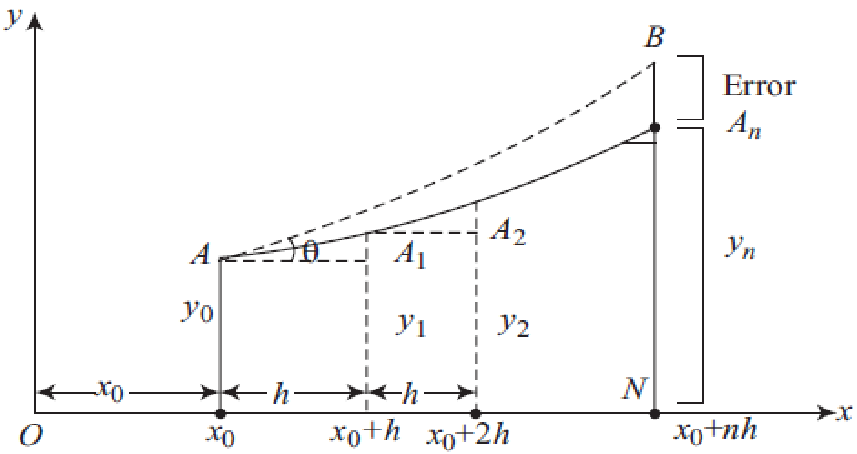



**Fig. 1**

Let $A\ (x_0, y_0)$ be the point on the curve of solution (I) and we wish to find the values of the curve from $x = x_0$ to $x = x_n$.

The curve of solution of (I) through $A(x_0, y_0)$ is as shown plotted in the **Figure 1**.

Here the interval $(x_0, x_n)$ is $MN$, so divide $MN$ into $n$-subintervals as $(x_0, x_0 + h)$, $(x_0 + h, x_0+2h)$, … so on. At the points $A(x_0, y_0)$, $A_1\ (x_0 + h, y_1)$, $A_2(x_0 + 2h, y_2)$, ... we approximate the curve of solution by the tangents at $A, A_1, A_2$ … whose slopes are $f(x_0, y_0), f(x_1, y_1), f(x_2, y_2)$, ... respectively. Therefore,

we get

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$y_3 = y_2 + h\,f(x_2, y_2)$$

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

In Euler's method, we approximate the curve of solution by the tangent line in each sub interval. If the slope of this tangent line changes rapidly over an interval, then its value at the beginning of the interval gives a poor approximation in comparison with its average value over the interval. Therefore, the value of $y$ has much error than its true value. This error accumulates in succeeding intervals and so the final value of $y$ has a lot of error. Hence, this method is quite useless.

**Example 1.** Given $y' = x + 2y$ with initial conditions $y(0) = 0$, find a value for the solution at $x = 1$, and the step size is $h = 0.25$. Find the solution by Euler's method.

**Solution:** Clearly the description of the problem implies that the interval for finding a solution will be [0,1]. The differential equation given tells us the formula for $f(x, y)$ required by the Euler method, namely:

$$f(x, y) = x + 2y$$

and the initial conditions tells us the values of the coordinates of our starting point,

$$x_0 = 0, y_0 = 0$$

We now use the Euler's method formulas to generate values for $x_1$ and $y_1$.

The formula, for '$x$' with $n = 0$ gives us:

$$x_1 = x_0 + h$$
$$x_1 = 0 + 0.25= 0.25$$

and the formula for '$y$' with $n = 0$ gives us:

$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_1 = y_0 + h(x_0+ 2y_0)$$
$$y_1 = 0 + 0.25 (0 + 2 * 0) = 0$$

Therefore $x_1 = 0.25$ and $y_1 = 0$.

Now move on to get the next point in the solution $(x_2, y_2)$ the $x$-iteration formula, with $n = 1$ give us:

$$x_2 = x_1 + h$$
$$x_2 = 0.25 + 0.25 = 0.5$$

And $y$-iteration formula with n = 1 gives us

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = y_1 + h(x_1 + 2y_1)$$
$$y_2 = 0 + 0.25 (0.25 + 2(0)) = 0.0625$$

Therefore $x_2 = 0.5$ and $y_2 = 0.0625$

We now move to get the fourth point in the solution $(x_3, y_3)$ the $x$-iteration formula, with $n = 2$ gives us:

$$x_3 = x_2 + h$$
$$x_3 = 0.5 + 0.25 = 0.75$$

and the $y$-iteration formula, with $n = 2$ gives us

$$y_3 = y_2 + hf(x_2, y_2)$$
$$y_3 = y_2 + h(x_2 + 2y_2)$$
$$y_3 = 0.0625 + 0.25 (0.5 + 2(0.0625)) = 0.21875$$

Therefore, $x_3 = 0.75$ and $y_3 = 0.21875$

Similarly $x_4 = 1$ and $y_4 = 0.515625$

We could summarize the results of all our calculations in a tabular form, as follows:

| $X$ | $Y$ |
|---|---|
| 0 | 0 |
| 0.25 | 0.03718 |
| 0.5 | 0.179570 |
| 0.75 | 0.495422 |
| 1 | 1.097264 |

We can get an even better feel for the inaccuracy. We have inaccuracy if we compare the graphs of the numerical and true solutions, as shown here:

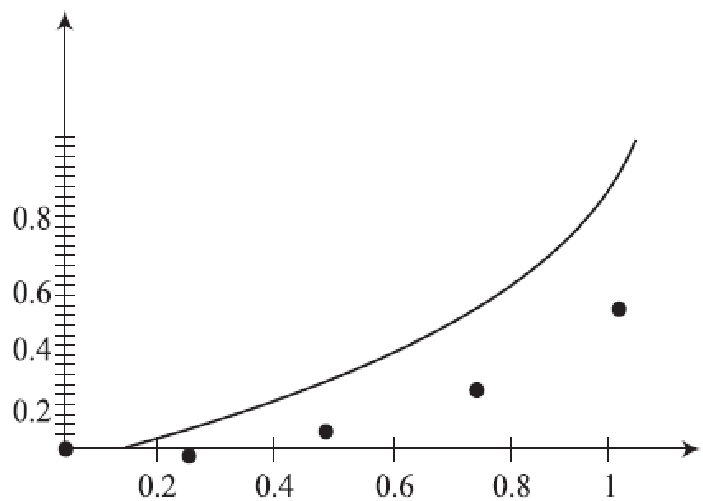



**Fig. 2**

The numerical solution gets worse and worse as we move further to the right, we might even be promoted to ask the question "what good is a solution that is this bad?" the answer is "little good at all" So should we quit using this method? No! The reason our numerical solution is so inaccurate is because the step size is so large. To improve the solution, shrink the step size.

**Example 2.** Given $\frac{dy}{dx} + 0.4y = 3e^{-x}$; $y(0) = 5$ and $h = 1.5$ find $y(3)$ using Euler's method.

**Solution:** Here $x_0 = 0$ and $y_0 = 5$

When $n = 0$

$$y_1 = y_0 + f(x_0, y_0)h$$
$$x_0 = 0, y_0 = 5, h = 1.5$$
$$y_1 = 5 + f(0, 5)\,(1.5)$$
$$= 5 + (3e^{-0} - 0.4\,(5)\,)(1.5) = 5 + (1)\,(1.5) = 6.5$$

Therefore $y_1 = 6.5$

When $n = 1$

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_1 = 1.5, y_1 = 6.5, h = 1.5$$
$$y_2 = 6.5 + f(1.5, 6.5)\,(1.5) \text{ as } (x_2 = x_1 + h = 1.5 + 1.5 = 3)$$
$$= 6.5 + (3e^{-1.5} - 0.4\,(6.5))\,(1.5)$$
$$= 6.5 + (-1.93061)\,(1.5) = 3.604$$

Therefore $y_2 = 3.604$ when $x_2 = 3$

$\Rightarrow$ $y(3) = 3.604.$

**Example 3.** Use Euler's method with step size 0.3 to compute the approximate $y$-value $y(0.9)$ of the solution of the initial value problem $y' = x^2$, $y(0) = 1$.

**Solution:** Here $x_0 = 0$, $y_0 = 1$ and $h = 0.3$

$$y_1 = y_0 + hf(x_0, y_0) \text{ as } (x_1 = x_0 + h = 0 + 0.3 = 0.3)$$

$$y_1 = 1 + 0.3\ (0) = 1$$

Therefore $x_1 = 0.3;\ y_1 = 1$

$$y_2 = y_1 + 0.3f(x_1, y_1) \text{ as } (x_2 = x_1 + h = 0.3 + 0.3 = 0.6)$$

$$y_2 = 1 + 0.3\ (0.3)^2 = 1.027$$

Therefore $y_2 = 1.027;\ x_2 = 0.6$

$$y_3 = y_2 + 0.3\ f(x_2, y_2) \text{ as } (x_3 = x_2 + h = 0.6 + 0.3 = 0.9)$$

$$= 1.027 + 0.3\ (0.6)^2 = 1.135$$

$$\Rightarrow \qquad y_3 = 1.135;\ x_3 = 0.9$$

$$\Rightarrow \qquad y(0.9) = 1.135.$$

**Example 4.** The exact solution to the initial value problem $y' = y$, $y(0) = 1$ is $y = e^x$. What will be the error when computing $y(3)$ using Euler's method with a step size of 1?

**Solution:** Here $x_0 = 0;\ y_0 = 1;\ h = 1,\ f(x, y) = y$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 1\ (1) = 2$$

$$y_2 = y_1 + hf(x_1, y_1) = 2 + 1\ (2) = 4$$

$$y_3 = y_2 + hf(x_2, y_2) = 4 + 1\ (4) = 8$$

$$y(3) = 8$$

Here $y = e^3 = 20.086 =$ Exact value

Therefore error $= 12.086$.

## Modified Euler's Method

In modified Euler's method, we start with the initial value of $y_0$, and approximate value of $y_1$ is calculated from Euler's method which gives the first approximation of $y_1$ at $x = x_1$ to this method. This first approximation is given by-

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

Consider the first interval $(x_0, x_1)$. Now at the end of this interval, we get

$$\left(\frac{dy}{dx}\right)_1^{(1)} = F(x_1, y_1^{(1)}) \qquad \left[\text{Therefore } \frac{dy}{dx} = F(x, y)\right]$$

Therefore, the improved value of $\Delta y$ is given by $\Delta y = h$

[Average value of $\frac{dy}{dx}$ in the interval $(x_0, x_1)$]

$$= h \cdot \frac{1}{2}\left[\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(1)}\right]$$

$$= \frac{1}{2}h[F(x_0, y_0) + F(x_1, y_1^{(1)})]$$

Now the second approximation of $y$ is given

$$y_1^{(2)} = y_0 + \Delta y = y_0 + \frac{1}{2}h[F(x_0, y_0) + F(x_1, y_1^{(1)})]$$

Again, the same interval $(x_0, y_0)$, we get an approximate value of $\frac{dy}{dx}$ at $x_1$ by putting the improved value of

$y_1^{(2)}$ of $y$ in $\frac{dy}{dx} = f(x, y)$, which is given by

$$\left(\frac{dy}{dx}\right)^{(2)} = F(x_1, y_1^{(2)})$$

Similarly, the third approximation of $y_1$ is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(2)}\right]$$

$$= y_0 + \frac{h}{2}[F(x_0, y_0) + F(x_1, y_1^{(2)})]$$

This process continues till two consecutive approximate value of $y$ at $x_1$. Then taking as the starting point for the next interval $(x_1, x_2)$ once $y_1$ is obtained to desired degree of accuracy, thenthe value of $y$ corresponding to $x_2 = x_0 + 2h$ is found by Euler's method, which is given by

$$y_2^{(1)} = y_1 + hF(x_1, y_1)$$

which gives then the first approximation of $y_2$.

Now repeat the above process to find the better approximation of $y_2^{(1)}$, $y_2^{(2)}$, $y_2^{(3)}$, etc.

In this same way the first approximations of $y_3, y_4 \ldots$ etc could be found with the help of Euler's method and then repeat above process to find better approximation of $y_3, y_4 \ldots$ etc.

**Example 5.** Given equation $\frac{dy}{dx} = x + y$ where $y(0) = 1$. Find the value of $y$ when $x = 1$ by using Euler modified method upto 3 decimal places taking $h = 0.5$.

**Solution:** First by using Euler method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 1 + 0.5\ (0 + 1) = 1.5$$

By using Euler modified method

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

Therefore, $x_1 = x_0 + h = 0.5$

$$y_1^{(1)} = 1 + \frac{0.5}{2}[(0 + 1) + (0.5 + 1.5)] = 1.75$$

$$y_1^{(2)} = 1 + \frac{0.5}{2}[(0 + 1) + (0.5 + 1.75)] = 1.8125$$

$$y_1^{(3)} = 1 + \frac{0.5}{2}[(0 + 1) + (0.5 + 1.8125)] = 1.828125$$

$$y_1^{(3)} = 1 + \frac{0.5}{2}[(0 + 1) + (0.5 + 1.8125)] = 1.828125$$

$$y_1^{(4)} = 1 + \frac{0.5}{2}[(0 + 1) + (0.5 + 1.828125)] = 1.83203125$$

$$y_1^{(5)} = 1 + \frac{0.5}{2}[(0{+}1\ ) + (0.5 + 1.8320312)] = 1\ .833007813$$

$$y_1^{(6)} = 1 + \frac{0.5}{2}[(0{+}1\ ) + (0.5 + 1.833007813)] = 1.833251953$$

Now as we can see that there is a minor difference in 5th and 6th iteration

Therefore, $y_1 = 1.833$

for $y_2 = y_1 + h\,[f(x_1, y_1)],\ y_1 = 1.833,\ x_1 = 0.5$

$$y_2 = 1.833 + 0.5\ (0.5 + 1.833)$$

$$y_2 = 2.9995$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1\ , y_1) + f(x_2\ , y_2)],\ x_2 = x_1 + h = 1$$

$$y_2^{(1)} = 1.833 + \frac{0.5}{2}[(0.5 + 1.833) + (1 + 2.995)] = 3.415$$

$$y_2^{(2)} = \mathrm{y}_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2\ , y_2^{(1)})]$$

$$= 1.833 + \frac{0.5}{2}[(0.5 + 1.833) + (1 + 3.415)] = 3.520$$

$$y_2^{(3)} = 1.833 + \frac{0.5}{2}[(0.5 + 1.833) + (1 + 3.520)] = 3.54625$$

$$y_2^{(4)} = 1.833 + 0.25\ [2.333 + (1 + 3.54625)] = 3.5528125$$

$$y_2^{(5)} = 1.833 + 0.25\ (2.333 + (1 + 3.5531125)) = 3.55445312$$

$$y_2^{(6)} = 1.833 + 0.25\ (2.333 + (1 + 3.55445312))$$

$$y_2^{(6)} = 3.55486328$$

As the difference between 5th iteration and 6th iteration is very minor

Therefore, $y_2 = 3.554$ and so $y(1) = 3.554$.

**Example 6.** Given $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$. Find an approximate value of $y$ at $x = 0.2$ by Euler's modified method taking step size $h = 0.1$.

**Solution:** First by using Euler method

$$y_1 = y_0 + hf\,(x_0, y_0)$$

Here $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$y_1 = 1 + 0.1\ (0 + 1)$$

$$= 1 + 0.1\ (1) = 1.1$$

Now $x_1 = x_0 + h$

$$x_1 = 0 + 0.1 = 0.1$$

By modified Euler method

$$y_1^{(1)} = y_0 + \frac{h}{2}[\mathrm{f}(\mathrm{x}_0\ , \mathrm{y}_0) + \mathrm{f}(\mathrm{x}_1\ , \mathrm{y}_1)]$$

$$= 1 + 0.05[1 + 0.1 + (1.1)^2] = 1.1155$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + 0.05[1 + 0.1 + (1.1155)^2] = 1.117$$

So, $y_1$ is 1.117 after 2nd iteration.

$$y_2 = y_1 + h\,[f(x_1, y_1)]$$

$$= 1.117 + 0.1(0.1 + (1.117)^2) = 1.23438$$

$$y_2^{(1)} = y_1 + h/2\,[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.117 + 0.05\,[0.1 + 1.247689 + 0.2 + (1.23438)^2] = 1.2705$$

So, $y_2$ is 1.2705 after 2nd iteration.

Therefore $y(0.2) = 1.2705$.

**Example 7.** Find the solution of the following differential equation by Euler's modified method for $x = 0.05$ and $x = 0.1$ by taking $h = 0.05$ correct upto 3 decimal places

$$\frac{dy}{dx} = x + y, \qquad [y = 1 \text{ when } x = 0]$$

**Solution:** Given $x_0 = 0$, $y_0 = 1$

$$x_1 = 0 + 0.05 = 0.05$$

$$x_2 = 0.05 + 0.05 = 1$$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05\,f(0, 1)$$

$$= 1 + 0.05\,[0 + 1] = 1.05$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.05}{2}[f(0\,,1) + f(0.05, 1.05)] = 1.0525$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + 0.05/2\,[f(0, 1) + f(0.05, 1.0525)] = 1.0525$$

Therefore, $y_1 = 1.0525$

Now $y_1 = 1.0525$; $x_1 = 0.05$, $x_2 = 0.1$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1)$$

$$= 1.0525 + 0.05\,f(0.05, 1.0525)$$

$$= 1.0525 + 0.05\,[0.05 + 1.0525] = 1.108$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.0525 + \frac{0.05}{2}[f(0.05, 1.0525) + f(0.1\,, 1.108)] = 1.1102$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

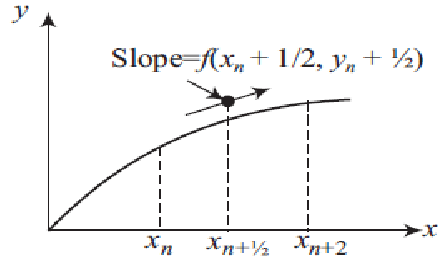
$$= 1.0525 + \frac{0.05}{2}[f(0.05, 1.0525) + f(0.1, 1.1102)] = 1.1103$$

$$\Rightarrow \qquad y_2 = 1.1103$$

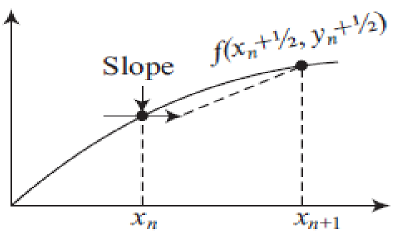
Therefore, $y(0.05) = 1.0525$, $y(0.1) = 1.1103$.

**Mid-Point Method**

Mid-point method is a simple modification of Euler's method. The mid-point method is a one-step method for numerically solving the differential equations. Midpoint method uses Euler's method to predict value of $y$ at mid-point of interval. The predicted value is used to calculate slope at the mid-point. Graphical representation of mid-point method




This is predictor graph




This is corrector graph

In this method

$$y_{n+1} = y_n + F\left[x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}F(x_n, y_n)h\right]h$$

This midpoint method is superior to Euler's method because it utilizes a slope estimate at the midpoint of the prediction interval.

**Example 8.** Given $\frac{dy}{dx} = y^2 + 5x$ when $y(0) = 2$. Taking $h = 0.5$ find the value of $y(1.5)$ using mid point method.

**Solution:** Here, $x_0 = 0;\ y_0 = 2;\ h = 0.5$

$$x_1 + x_0 + h = 0 + 0.5 = 0.5$$

First calculate

$$F(x_0, y_0) = {y_0}^2 + 5x_0 = 4$$

and $$x_0 + \frac{h}{2} = 0 + \frac{0.5}{2} = 0 + 0.25 = 0.25$$

$$\begin{aligned} y_1 &= y_0 + F\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}F(x_0, y_0)h\right]h \\ &= 2 + F[0.25, 2 + 2(0.5)]0.5 \\ &= 2 + F[0.25, 3]0.5 \\ &= 2 + (3^2 + 5(0.25))0.5 = 7.125, \end{aligned}$$

therefore $y(0.5) = 7.125$

$$F(x_1, y_1) = {y_1}^2 + 5x_1 = 53.265625$$

$$\begin{aligned} y_2 &= y_1 + F\left[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}F(x_1, y_1)h\right]h \\ &= 7.125 + F[0.5 + 0.25, 7.125 + 13.3164062]0.5 \\ &= 7.125 + F[0.75, 20.4414062]0.5 = 217.925544 \end{aligned}$$

Therefore $$y(1) = 217.925544$$

$$f(x_2, y_2) = y_2^2 + 5x_2 = 47496.5427$$

$$\begin{aligned} y_3 &= y_2 + F\left[x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}F(x_2, y_2)h\right]h \\ &= 217.925544 + \mathrm{F}[1.25,12092.0612]0.5 = 73109193.1 \end{aligned}$$

$$F(x_3, y_3) = y_3^2 + 5x_3 = 5.34495e15.$$

**Example 9.** Given $\frac{dy}{dx} = x + y$; $y(0) = 1$ and $h = 0.05$; find $y(0.05)$ using Mid point method.

**Solution:** Given

$$\begin{aligned} x_0 &= 0, y_0 = 1, h = 0.05 \\ x_1 &= x_0 + h = 0 + 0.05 = 0.05 \\ F(x_0, y_0) &= x_0 + y_0 = 0 + 1 = 1 \\ x_0 + \frac{h}{2} &= 0 + \frac{0.05}{2} = 0.025 \end{aligned}$$

Therefore, $$\begin{aligned} y_1 &= y_0 + F\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}F(x_0, y_0)h\right]h \\ &= 1 + F[0.025, 1.025]\ 0.05 \\ &= 1 + 0.05(1.05) = 1.0525 \end{aligned}$$

$$\Rightarrow \quad y(0.05) = 1.0525.$$

**Example 10.** Given $\frac{dy}{dx} = x + y^2$; $y(0) = 1$ and $h = 0.1$ find $y(0.1)$ using mid point method.

**Solution:** Given

$$x_0 = 0;\ y_0 = 1;\ h = 0.1$$

$$x_1 = x_0 + h = 0.1$$

$$F(x_0, y_0) = F(0, 1) = 1$$

$$x_0 + \frac{h}{2} = 0.05$$

$$y_1 = y_0 + F\left[x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}F(x_0, y_0)h\right]h$$

$$= 1 + F[0.05,\ 1 + 0.05]0.1 = 1 + F[0.05,\ 1.05]0.1$$

$$= 1 + [0.05 + (1.05)^2]0.1 = 1.11525$$

$$\Rightarrow \qquad y(0.1) = 1.11525.$$