Linear Transformations

Course Name : B.Sc.(H) Physics (I Year, II Semester)
Paper Name:- Linear Algebra (1GE1)
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April 22, 2020

1 INTRODUCTION TO LINEAR TRANSFORMATIONS

In this section we introduce linear transformation and examine their elementary properties.

Definition (Linear Transformation) :

Let V and W be two vector spaces over $\mathbf{R}$. A function $T : V \to W$ is called a linear transformation from V to W if it satisfies the following properties:

(1) $T(v_1 + v_2) = T(v_1) + T(v_2)$, for all $v_1, v_2 \in V$.

(b) $T(\alpha v) = \alpha T(v)$, for all $\alpha \in \mathbb{R}$ and all $v \in V$.

Thus, a linear transformation is a function between two vector spaces that preserves the operations that give structure to the spaces.

To determine whether a given function from a vector space to another vector space is linear transformation, we need only verify properties (1) and (2) in the definition, as in the next examples

Example 1. Zero Linear Transformation Let V and W be two vector spaces. Consider the mapping $T : V \to W$ defined by $T(v) = 0_W$, for all $v \in V$. We will show that T is a linear transformation.
1. we must that $T(v_1 + v_2) = T(v_1) + T(v_2)$, for qall $v_1, v - 2 \in V$
Now $T(v_1 + v_2) = 0_w = 0_w + 0_w = T(v_1) + T(v_2)$.
2. we must show that $T(\alpha v) = \alpha T(v)$, for all $\alpha \in \mathbb{R}$ and for all $v \in V$.

Hence, T is a linear transformation.

Example 2. Consider the mapping $f : M_{mn} \to M_{mn}$, given by $f(A) = A^T$ for any $m \times n$ matrix A. we will show that f is a linear transformation

(1) We must show that $f(A_1 + A_2) = f(A_1) + f(A_2)$, for matrices $A_1, A_2 \in M_{mn}$. However, $f(A_1) + f(A_2) = (A_1 + A_2)^T = A_1^T + A_2^T = f(A_1) + f(A_2)$.

(2.) We must show that $f(cA) = cf(A)$, for all $A \in M_{mn}$ and for all $c \in \mathbb{R}$. However $f(cA) = (cA)^T = c(A)^T = cf(A)$.

Exercise 1. Let $V = P_n$, the space of all polynomial of degree $\leq n$, with real coefficient and $W = p_{n-1}$, the space of all polynomial of degree $\leq n-1$, with real coefficients. Consider the mapping $g : V \to W$ defined by

$$g(p) = P'$$

for any $p \in V$. Show that g is a linear transformation.

Definition (Linear Operator) : Let V be a vector space. A linear transformation $T : V \to V$ is called a linear operator. Thus, a linear operator is a linear transformation from a vector space to itself.

Exercise 2. Let V be a vector space. Consider the mapping $T : V \to V$ defined by $T(v) = v, \forall v \in V$. Then show that T is a linear operator.

Note 1. This linear operator T is called Identity Linear Operator.

Exercise 3. Let $k \in \mathbb{R}$. Define that mapping $f\mathbb{R}^n \to \mathbb{R}^n$ as $f(v) = kv, \forall v \in \mathbb{R}^n$. Then show that this f is linear operator.

Note 2. This f is called dilation or contraction, according as $|k| > 1$ or $|k| < 1$, respectively. If $|k| > 1$,f stretches the length of the vector and if $|k| < 1$, f shrinks the length.

Exercise 4. Consider the mapping $h : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $h([x_1, x_2, x_3]) = [x_1, x_2, 0]$, $[x_1, x_2, x_3] \in \mathbb{R}^3$. Then show that h is linear operator. This linear operator p is called projection.

Note 3. Notice that we can also define a projection on $\mathbb{R}^3$ which projects each vector in $\mathbb{R}^3$ to a corresponding vector in the $yz - plane$ or the $zx - plane$.

Exercise 5. Show that the mapping $r : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $r([x_1, x_2, x_3] = [x_1, x_2, x_3]$, $[x_1, x_2, x_3] \in \mathbb{R}^3$ is linear operator. This linear operator r is called Reflection operator.

Note 4. Notice that we can also define a reflection operator on $\mathbb{R}^3$ which reflects each vector in $\mathbb{R}^3$ through the $yz - plane$ or the $zx - plane$.

Example 3. Let θ be a fixed angle, and let $l : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$l\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$

Then show that l is a linear operator.

Solution Let $v_1 = [x-1, y_1]$ and $v_2 = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$. Then

$$l(v_1 + v_2) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} (v_1 + v_2)$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} v_1 + \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} v_2 = l(v_1) + l(v_2)$$

Similarly, $l(\alpha v) = \alpha l(v)$. Hence, l is a linear operator.

Exercise 6. Let A be a fixed $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(x) = Ax, \forall x \in \mathbb{R}^n$$

show that T is a linear transformation.

Exercise 7.(Shear Operator) Let k be a fixed scalar in $\mathbb{R}$. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x & \\ y & \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & \\ y & \end{bmatrix} = \begin{bmatrix} x+ky & \\ y & \end{bmatrix}$$

Show that T is a linear transformation.

Theorem 1 *(Properties of Linear Transformation)*

Let V and W be two vector spaces, and let $L : V \to W$ be a linear transformation. Let 0_v be the zero vector in V and 0_W be the zero vector in W. Then

1. $l(o_v) = 0_W$

2. $L(-v) = -L(v), \forall v \in V$

3. $L(u - v) = L(u) - L(v), \forall u, v \in V$

4. $L(a_1v_1+a_2v_2+....+a_nv_n) = a_1L(a_1)+a_2L(v_2)+...+a_nL(a_n), \forall a_1, a_2, ..., a_n \in \mathbb{R}, v_1, v_2, .., v_n \in V$,*for* $n \geq 1$

Proof 1.

$$L(0_V) = L(00_V) = 0L(0_V) = 0_W.$$

Hence, 1. is proved.

2.

$$L(-v) = L((-1)v) = (-1)L(v) = -L(v)$$

so, 2. is proved.

3.

$$L(u - v) = L(u + (-1)v) = L(u) + L((-1)v) = L(U) - L(v).$$

. **4.** This part is proved by induction.
For $n = 1$, we have $L(a_1v_1) = a_1L(v_1)$ (Property (1) of linear transformation) so the result is true for $n = 1$.
Similarly for $n = 2$, we have
$L(a_1v_1 + a_2v_2) = L(a_1v_1) + L(a_2v_2)$ (by property (1) of linear transformation) $= a_1L(v_1) + a_2L(v_2)$ (Property (2) of linear transformation) so, the result is true for $n = 2$.
Now we will assume that the reslut is true for $n = m$ i.e.,

$L(a_1v_1 + a_2v_2 + ... + a_mv_m) = a_1L(v_1) + ... + a_mL(v_m)$

We have

$$L(a_1v_1+a_2v_2+...+a_mv_m+a_{m+1}v_{m+1}) = L(a_1v_1+a_2v_2+...+a_mv_m)+L(a_{m+1}v_{m+1})$$

$$\text{(property (2) of linear transformation)}$$

$$= a_1L(v_1) + ... + a_mL(v_m) + a_{m+1}v_{m+1}$$

(by the induction hypothesis)

So, the result is true for $n = m + 1$. Hence, by the principle of mathematical induction, the result is true for any natural number n.

Example 4. Let V be a vector space and let $x \neq 0$ be a fixed vector in V. Prove that the translation function $f : V \to v$ defined by $f(v) = v + x$ is not a linear transformation.

Solution We have

$$f(0) = 0 + x = x \neq 0$$

but by part (1) of previous theorem we know that any linear transformation sends identity element to identity element. Therefore, f is not a linear transformation.

Composition of Linear Transformation If $f : X \to Y$ and $g : Y \to Z$ are functions, then the composition of f and g is defined to be the function $g \circ f : X \to Z$ given by $(g \circ f) = g(f(x))$. The followng theorem says that the composition of linear transformations is again a linear transformation

Theorem 2 *Let V_1, v_2 and V_3 be vector spaces. Let $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ be linear transformations. Then $L_2 \circ L_1 : V_1 \to V_3$ given by $(L_2 \circ L_1)(v) = L_2(L_1(v))$, for all $v \in V_1$, is a linear transformation.*

Proof To show that $L_2 \circ L_1$ is a linear transformation, we must show that for all $c \in \mathbb{R}$ and $v, v_1, v_2 \in V$

$$(L_2 \circ L_1)(v_1 + v_2) = (L_2 \circ L_1)(v_1) + (L_2 \circ L_1)(v_2)$$

$$(L_2 \circ L_1)(cv) = c(L_2 \circ L_1)(v)$$

The first property holds since

$$(L_2 \circ L_1)(v_1 + v_2) = L_2(L_1(v_1 + v_2))$$

$$= L_2(L_1(v_1) + L_1(v_2))$$

(because L_1 is a linear transformation)

$$= L_2(L_1(v_1)) + L_2(L_1(v_2))$$

(Because L_2 is a linear transformation)

So the property first is hold.
To prove the second property, consider

$$(L_2 \circ L_1)(cv) = L_2(L_1(cv))$$

(Definition of composition)

$$= L_2(L_1(cv))$$

(Since L_1 is a Linear Transformation)

$$= L_2(cL_1(v)) = c(L_2(L_1(v))$$

$$= c(L_2 \circ L_1)(v)$$

So the second property also holds.
Hence, $L_2 \circ L_1$ is a linear transformation.

Example 5. Let L_1 represent the rotaion of vecotrs in $\mathbb{R}^2$ through a fixed angle θ and let L_2 represent the reflection of vectors in $\mathbb{R}^2$ through the $x - axis$. That is if $v = [v_1, v_2]$, then
$L_1(v) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $L_2(v) = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$

Because L_1 and L_2 are both linear transformations, theorem 2 says that

$$L_2(L_1(v)) = L_2\left(\begin{bmatrix} v_1\cos(\theta) - v_2\sin(\theta) \\ v_1\sin(\theta) + v_2\cos(\theta) \end{bmatrix}\right) = \begin{bmatrix} v_1\cos(\theta) - v_2\sin(\theta) \\ -v_1\sin(\theta) - v_2\cos(\theta) \end{bmatrix}$$

is also a linear transformation, $L_2 \circ L_1$ represents a rotation of v through θ followed by a reflection through the $x - axis$.

Note 5. Theorem 2. generalizes naturally to more than two linear transformation. i.e. if $L_1, L_2, ..., L_k$ are linear transformation and the composition $L_1 \circ L_2 \circ ... \circ L_k$ makes sense. then $L_k \circ ... \circ L_2 \circ L_1$ is also a linear transformation.

Linear Transformation and Subspaces The final theorem of this section assures us that, under a linear transformation $L : V \to W$, subspaces of V are mapped to subspace of W and vice vera.

Definition (Image and Pre-Image)

LetV_1 and V_2 be vector spaces and let $T : V_1 \to V_2$ be a linear transformation. Given any set $U \subseteq V_1$, the image of U in V_2 is defined to be the set

$$T(U) = \{T(U) : u \in U\}$$

Similarly, given a set $W \subseteq V_2$, the pre-image of W in V_1 is defined to be the set

$$T^{-1}(W) = \{v \in V_! : T(v) \in W\}$$

Theorem 3 *Let $L : V \to W$ be alinear transformation*

(1)If V' is a subspace of V, then $L(V')$, the image of V' in W is a subspace of W. In particular, the range of L is subspace of W.

(2) If W' is a subspace of W, then $L^{-1}(W')$, the pre-image of W' in V, is a subspace of V.

Proof 1. Suppose that $L : V \to W$ is a linear transformation and that V' is a subspace of V. Since, V' is a subspace of V then $0_V \in V'$. and we know that any linear transformations maps zero element to zero element. Thus,

$$0_W = L(0_V) \in L(V').$$

Hence, $L(V')$ is non-empty. Therefore to show that $L(V')$ is a subspace of W, we must show that $L(V')$ is closed under addition and scalar multiplication.

First suppose that $w_1, w_2 \in L(V')$. Then by definition of $L(V')$, we have $w_1 = L(v_1)$ and $w_2 = L(v_2)$, for some $v_1, v_2 \in V'$. Then, $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$ because L is a linear transformation. However, since V' is a subspace of V, $(v_1 + v_2) \in V'$. Thus, $(w_1 + w_2)$ is the image of $(v_1 + v_2) \in V'$, and so $(w_1 + w_2) \in L(V')$. Hence, $L(V')$ is closed under addition.

Next, suppose that $c \in \mathbb{R}$ and $w \in L(V')$. By definition of $L(V')$, $w = L(v)$ for some $v \in V'$. Then, $cw = cL(v) = L(cv)$ since L is linear transformation. Now, $cv \in V'$, because V' is a subspace of V. Thus, cw is the image of $cv \in V'$ and so $cw \in L(V')$. Hence, $L(V')$ is closed under scalar multiplication. **2.** The

pre-image of a subspace W' of W is given by $L^{-1}(W') = \{v \in V : L(v) \in W'\}$
$\because$ $L(0_V) = 0_W \in W'$, so $0_V \in L^{-1}(W;)$ $\therefore$ L^{-1} is non-empty.
Also,let $v_1, v_2 \in L^{-1}(W') \implies L(v_1), L(v_2) \in W' \implies L(v_1) + L(v_2) \in W' \implies L(v_1 + v_2) \in W' \implies v_1 + v_2 \in L^{-1}(W')$
Finally, let $v \in L^{-1}(W')$, and let $c \in \mathbb{R}$. Then

$$v \in L^{-1}(W') \implies L(v) \in W' \implies cL(v) \in W' \implies L(cv) \in W' \implies cv \in L^{-1}(W')$$

Hence, $L^{-1}(W')$ is a subspace of V.

2 The Matrix Of A Linear Transformation

In this section, we will show that the behavior of any linear transformation $T : V \to W$ is determined by the effect on a basis for V. In particular, when V and W are finite dimensional and ordered bases for V and W are chosen we can obtain a matrix corresponding to T that is useful in computing images under T.

A Linear Transformation is Determined by its Action on a Basis

One of the most useful properties of linear transformations is that, if we know how a linear map $T : V \to W$ acts on a basis of V,then we known how it acts on the whole of V.

Theorem 4 *Let $B = \{v_1, v_2, ..., v_n\}$ be an ordered basis for a vector space V. Let W be a vector space, and let $\{w_1, w_2, ..., w_n\}$ be any n (not necessarily distinct) vectors in W. Then there is one and only one linear transformation $T : V \to W$ satisfying $T(v_1) = w_1, T(v_2) = w_2, ..., T(v_n) = w_n$. In other words, a linear transformation is determined by its action on a basis.*

Proof Let v be any vector in V. Since $B = \{v_1, v_2, ..., v_n\}$ is an ordered basis for V, there exist unique scalars $a_1, a_2, ..., a_n$ in $\mathbb{R}$ such that $v = a_1v_1 + a_2v_2 + ... + a_nv_n$. Define a function $T : V \to W$ by

$$T(v) = a_1w_1 + a_2w_2 + ... + a_nw_n$$

Since the scalars $a_i's$ are unique, T is well defined. We will show that T is a linear transformation. Let x and y be two vectors in V. Then

$$x = b_1v_1 + b_2v_2 + ... + b_nv_n$$

and

$$y = c_1v_1 + c_2v_2 + ... + c_nv_n$$

for some unique $b_i's$ and $c_i's$ in $\mathbb{R}$. Then by definition of T, we have

$$T(x) = b_1w_1 + b_2w_2 + ... + b_nv_n$$

$$T(y) = c_1w_1 + c_2w_2 + ... + c_nw_n$$

$$\begin{aligned} T(x) + T(y) &= (b_1w_1 + b_2w_2 + ... + b_nw_n) + (c_1w_1 + c_2w_2 + ... + c_nw_n) \\ &= (b_1 + c_1)w_1 + (b_2 + c_2)w_2 + ... + (b_n + c_n)w_n \end{aligned}$$

However,

$$x + y = (b_1v_1 + b_2v_2 + ... + b_nv_n) + (c_1v_1 + c_2v_2 + ... + c_nv_n)$$

$$= (b_1 + c_1)v_1 + (b_2 + c_2)v_2 + ... + (b_n + c_n)v_n$$

$$T(x + y) = (b_1 + c_1)w_1 + (b_2 + c_2)w_2 + ... + (b_n + c_n)w_n$$

again by definition of T. Hence, $T(x+y) = T(x) + T(y)$. Next, for any scalar $c \in \mathbb{R}$,

$$cx = c(b_1v_1 + b_2v_2 + ... + b_nv_n) = (cb_1)v_1 + (cb_2v_2) = ... + (cb_n)v_n$$

$\implies$

$$\begin{aligned} T(cx) &= (cb_1)w_1 + (cb_2)w_2 + ... + (cb_n)w_n \\ &= c(b_1w_1) + c(b_2w_2) + ... + c(b_nw_n) \\ &= c(b_1w_1 + b_2w_2 + ... + b_nw_n) \\ &= cT(x) \end{aligned}$$

Hence T is a linear transformation. To prove the uniqueness, let $L : V \to W$ be another linear transformation satisfying

$$L(v_1) = w_1, L(v_2) = w_2, ..., L(v_n) = w_n$$

If $v \in V$, then $v = a_1v_1 + a_2v_2 + ... + a_nv_n$, for unique scalars $a_1, a_2, ..., a_n \in \mathbb{R}$.
But then

$$\begin{aligned} L(v) &= L(a_1v_1 + a_2v_2 + ... + a_nv_n) \\ &= a_1L(v_1) + a_2L(v_2) + ... + a_nL(v_n) \qquad (L \;\; is \;\; a \;\; L.T) \\ &= a_1w_1 + a_2w_2 + ... + a_nw_n = T(v) \end{aligned}$$

$\implies$ $L = T$ and hence T is uniquely determined.

Example 6. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with $L([1,-1,0])=$ [2,1],$L([0,1,-1])=$ [-1,3] and $L([0,1,0]) = [0,1]$. Find $L([-1,1,2])$. Also, give a formula for $L([x,y,z])$, for any $[x,y,z] \in \mathbb{R}^3$.

Solution To find $L[-1,1,2]$, we need to express the vector $v = [-1,1,2]$ as a linear combination of vectors $v_1 = [1,-1,0]$, $v_2 = [0,1,-1]$ and $v_3 = [0,1,0]$. That is, we need to find constants a_1, a_2 and a_3 such that

$$v = a_1v_1 + a_2v_2 + a_3v_3$$

, which leads to the linear system whose augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 : -1 \\ -1 & 1 & 1 : 1 \\ 0 & -1 & 0 : 2 \end{pmatrix}$$

we transform this matrix to reduced row echelon form :

$$\begin{pmatrix} 1 & 0 & 0 : -1 \\ -1 & 1 & 1 : 1 \\ 0 & -1 & 0 : 2 \end{pmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{pmatrix} 1 & 0 & 0 : -1 \\ 0 & 1 & 1 : 0 \\ 0 & -1 & 0 : 2 \end{pmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 0 & 0 : -1 \\ 0 & 1 & 1 : 0 \\ 0 & 0 & 1 : 2 \end{pmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{pmatrix} 1 & 0 & 0 : -1 \\ 0 & 1 & 0 : -2 \\ 0 & 0 & 1 : 2 \end{pmatrix}$$

This gives $a_1 = -1, a_2 = -2$, and $a_3 = 2$. So,

$$v = -v_1 - 2v_2 + 2v_3$$

$\Longrightarrow$

$$L(v) = L(-v_1 - 2v_2 + 2v_3)$$

$$= L(-v_1) - 2L(v_2) + 2L(v_3)$$

$$= -[2, 1] - 2[-1, 3] + 2[0, 1] = [0, -5]$$

$i.e$

$$L([-1, 1, 2]) = [0, -5]$$

To find $L([x, y, z])$ for any $[x, y, z] \in \mathbb{R}^3$, we row reduce

$$\begin{pmatrix} 1 & 0 & 0 : x \\ -1 & 1 & 1 : y \\ 0 & -1 & 0 : z \end{pmatrix}$$

to obtain

$$\begin{pmatrix} 1 & 0 & 0 : x \\ 0 & 1 & 0 : -z \\ 0 & 0 & 1 : x + y + z \end{pmatrix}$$

Thus,

$$[x, y, z] = xv_1 - zv_2 + (x + y + z)v_3$$

$$L([x, y, z]) = L(xv_1 - zv_2 + (x + y + z)v_3$$

$$= xL(v_1) - zL(v_2) + (x + y + z)L(v_3)$$

$$= x[2, 1] - z[-1, 3] + (x + y + z)[0, 1]$$

$$= [2x + z, 2x + y, -2z]$$

. **Exercise 8.** Suppose $L : \mathbb{R}^2 \to R^2$ is a linear operator and $L([1, 1]) = [1, -3]$ and $L([-2, 3]) = [-4, 2]$. Express $L([1, 0])$ and $L([0, 1])$ as linear combination of the vectors $[1, 0]$ and $[0, 1]$.

The Matrix of Linear Transformation Our next goal is to show that any linear transformation on a finite dimensional vector space can be expressed as a matrix multiplication. This will allow us to find the effect of any linear transformation by simply using matrix multiplication.

Let V and W be non-trivial vector spaces, with $dim(V) = n$ and $dim(W) = m$. Let $B = \{v_1, v_2, ..., v_n\}$ and $C = \{w_1, w_2, ..., w_m\}$ be ordered basis for V and W, respectively. Let $T : V \to W$ be a linear transformation. For each v in V, the coordinate vectors for v and $T(v)$ with respect to ordered basis B and C are $[v]_B$ and $[T(v)]_C$, respectively. Our goal is to find an $m \times n$ matrix $A = (a_{ij})(i \leq i \leq m; 1 \leq j \leq n)$ such that

$$A[v]_B = [T(v)]_C \tag{1}$$

holds for all vectors v in V. Since Equation (1) must hold for all vectors in V, it must hold, in particular, for the basis vectors in B, that is

$$A[v_1]_B = [T(v_1)]_C, A[v_2]_B = [T(v_2)]_C, ..., A[v_n]_B = [T(v_n)]_C \tag{2}$$

But $[v_1] = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$ $[v_2]_B = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ..., [v_n]_B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$ Therefore

$$A[v_1]_B = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}$$

$$A[v_2]_B = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix}$$

$$A[v_n]_B = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix}$$

Substituting these results into (2), we obtain

$$\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix} = [T(v_1)]_C, \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix} = [T(v_2)]_C, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix} = [T(v_n)]_C,$$

This shows that the successive columns of A are the coordinate vectors of $T(v_1), T(v_2), ..., T(v_n)$ with respect to the ordered basis C. Thus, the matrix A is given by

$$A = [[T(v_1)]_C, [T(v_2)]_C \quad [T(v_n)]_C]$$

We will call this matrix as the matrix of T relative to the basis B and C and will denote it by the symbol A_{BC} or $[T]_{BC}$. Thus,

$$A_{BC} = [[T(v_1)]_C \;\; [T(v_2)]_C \quad [T(v_n)]_C]$$

From (1), the matrix A_{BC} satisfies the property

$$A_{BC}[v]_B = [T(v)]_C, \forall v \in V$$

we have thus proved

Theorem 5 *Let V and W be non-trivial vector spaces, with $dim(V) = n$ and $dim(W) = m$. Let $B = \{v_1, v_2, .., v_n\}$ and $C = \{w_1, w_2, ..., w_m\}$ be ordered basis for V and W respectively. Let $T : V \to W$ be a linear transformation. Then there is a unique $m \times n$ matrix A_{BC} such that $A_{BC}[V]_B = [T(v)]_C, \forall v \in V$. Furthermore, for $1 \leq i \leq n$ the ith column of $A_{BC} = [T(v_i)]_C$.*

Example 7. Let $T : P_3 \to \mathbb{R}^3$ be the linear transformation given by $T(ax^3 + bx^2 + cx + d) = [4a - b + 3c + 3d, a + 3b - c + 5d, -2a - 7b + 5c - d]$. Find the matrix for T with respect to the standard basis $B = \{x^3, x^2, x, 1\}$ for P_3 and $C = \{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^3$.

Solution We substitute standard basis vector B into the given formula for T shows that $T(x^3) = [4, 1, -2]$, $T(x^2) = [-1, 3, -7]$, $T(x) = [3, -1, 5]$ and $T(1) = [3, 5, -1]$. Since we are using standard basis C for $\mathbb{R}^3$,

$$[T(x^3)] = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, [T(x^2)]_C = \begin{bmatrix} -1 \\ 3 \\ -7 \end{bmatrix}, [T(x)]_C = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, [T(1)]_C = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

Thus, the matrix of T with repect to the bases B and C is:

$$A_{BC} = [[T(x^3)]_C \quad [T(x^2)]_C \quad [T(x)]_C \quad [T(1)]_C] = \begin{bmatrix} 4 & -1 & 3 & 3 \\ 1 & 3 & -1 & 5 \\ -2 & -7 & 5 & -1 \end{bmatrix}$$

Exercise 9 Let $T : P_3 \to P_2$ be the linear transformation given by $T(p) = p'$, where $p \in P_3$. Find the matrix for T with respect to the standard bases for P_3

and P_2. use this matrix to calculate $T(4x^3 - 5X^2 + 6x - 7)$ by matrix multiplication.

Exercise 10 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the formula $T([x_1, x_2, x_3]) = [-2x_1 + 3x_3, x_1 + 2x_2 - x_3]$. Find the matrix for T with respect to the ordered basis $B = \{[1, -3, 2], [-4, 13, -3], [2, -3, 20]\}$ for $\mathbb{R}^3$ and $C = \{[-2, -1], [5, 3]\}$

Solution By the definition, the matrix A_{BC} of T with respect to the ordered basis B and C is given by $A_{BC}[[T(v_1]_C \;\; [T(v_2)]_C \;\; [T(v_3)]_C]$, where $v_1 = [1, -3, 2], v_2 = [-4, 13, -3]$ and $v_3 = [2, -3, 20]$ are the basis vector in B. Substituting each basis vector in B into the given formula for T shows that

$$T(v_1) = [4, -7], T(v_2) = [-1, 25], T(v_3) = [56, -24]$$

Next we must find the coordinate vector of each of these images in $\mathbb{R}^2$ with respect to the C basis. To do this, we use coordinatization method. Thus, we must row reduce matrix

$$[w_1 \;\; w_2 \;\; | \, T(v_1) \;\; T(v_2) \;\; T(v_3)],$$

where $w_1 = [-2, -1], w_2 = [5, 3]$ are the basis vector in C. Thus, we row reduce

$$\left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ -1 & 3 & -7 & 25 & -24 \end{array}\right]$$

To obtain

$$\left[\begin{array}{cc|ccc} 1 & 0 & -47 & 128 & -288 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right]$$

Hence $[T(v_1)]_C = \begin{bmatrix} -47 \\ -18 \end{bmatrix}$,$[T(v_2)]_C = \begin{bmatrix} 128 \\ 51 \end{bmatrix}$, $[T(v_3)]_C = \begin{bmatrix} -288 \\ -104 \end{bmatrix}$

$\therefore$ The matrix of T with respect to the bases B and C is $A_{BC} = \begin{bmatrix} -47 & 128 & -288 \\ -18 & 51 & -104 \end{bmatrix}$

Finding the New matrix for a linear transformation After a Change of Basis

Theorem 6 *(Proof Omitted) Let V and W be two no-trivial finite dimensional vector space with ordered basis B and C, respectively. Let $L : V \to W$ is a linear transformation with matrix A_{BC} with respect to bases B and C. Suppose that D and E are other ordered bases for V and W respectively. Let P be the transition matrix from B to D, and let Q be the transition matrix from C to E is given by $A_{DE} = QA_{BC}P^{-1}$*

Note This theorem held us to find how the matrix for a linear transformation changes when we change the bases for the domain and co-domain.

Example 11. Let $L : P_3 \to \mathbb{R}^3$ be the linear transformation given by $L(ax^3 + bx^2 + cx + d) = [c + d, 2b, a - d]$.

(a) Find the matrix A_{BC} for L with respect to the standard basis $B(for\ P_3)$ and $C(for\ \mathbb{R}^3$.

(b) use part (a) to find the matrix A_{DE} for L with respect to the standard bases $D = \{x^3 + x^2, x^2 + x, x + 1, 1\}$ for P_3 and $E = \{[-2, 1, -3], [1, -3, 0], [3, -6, 2]\}$ for $\mathbb{R}^3$.

Solution (a) Left as an Exercise.
(b) To find A_{DE} we make use of the following relationship:

$$A_{DE} = QA_{BC}P^{-1}$$

where P is the transition matrix from B to D and Q is the transition matrix from C to E. Since, P is the transition matrix from B to D. Also, the transition matrix P^{-1} from bases D to B is

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (verify\ it)$$

To find Q, we first find Q^{-1}, the transition matrix from E to C, which is the matrix whose columns are the vectors in E.

$$Q^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix} \quad (Verify)$$

$$\implies \qquad Q = (Q^{-1})^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}$$

Hence,

$$A_{DE} = QA_{BC}P^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}$$

3 LINEAR OPERATOR AND SIMILARITY

Recall from chapter Eigenvalue and Diagonalization "Two matrix A and B are said to be similar if there exist a non-singular $n \times n$ matrix P such that $P^{-1}AP = B$.

In this section we will show that any two matrices for the same linear operator with respect to different ordered basis are similar.

Theorem 7 *(Without Proof) Suppose L is a linear operator on a finite dimensional vector space V. If B is a basis for V, then there is some matrix A_{BB} for L with respect to B. Also, if C is another basis for V, then there is some matrix A_{CC} for L with respect to C. Let P be the transition matrix from B to C. Notice that by Theorem 6 we have $A_{BB} = P^{-1}A_{CC}P$. Ans so by the definition of similar matrices, A_{BB} and A_{CC} are similar.*

Note This theorem shows that any two matrices for the same linear operator with respect to different bases are similar. In fact, the converse is also true.

Example 12. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that perform a counterclockwise rotation through an angle $30°$. Find the matrix A for L with respect to the standard ordered basis for $\mathbb{R}^2$. Hence or otherwise find the matrix, for L with respect to the basis $C = \{[4, -3], [3, -2]\}$, similar to A.

Solution We know that if $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator representing the counterclockwise rotation of vectors in $\mathbb{R}^2$ though a fixed angle θ, then the matrix for L with respect to the standard ordered basis $B = \{e_1 = [1, 0], e_2 = [0, 1]\}$ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

In particular, the matrix A for the linear operator L that performs a counterclockwise rotation through an angle $30°$ with respect to the standard basis B for $\mathbb{R}^2$ is given by

$$A = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

To find the matrix, for L with respect to the basis $C = \{[4, -3], [3, -2]\}$, similar to A, we first find the transition matrix P from the standard basis B to the basis C. Now P^{-1} the transition matrix from C to B, is give by

$$P^{-1} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \implies P = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}$$

Hence the matrix A_{CC} for L with respect to the basis C, similar to A is

$$A_{CC} = PAP^{-1} = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{3} - 9 & -\frac{13}{2} \\ \frac{25}{2} & \frac{\sqrt{3}}{2} + 9 \end{bmatrix}$$

Matrix For the Composition of Linear Transformation

Our final final theorem for this section shows how to find the corresponding matrix for the composition of linear transformation.

Theorem 8 *Let V_1, V_2 and V_3 be nontrivial finite dimensional vector spaces with ordered basis B, C and D respectively. Let $L_1 : V_1 \to V_2$ be a linear transformation with matrix A_{BC} with respect to bases B and C, and let $L_2 : V_2 \to V_3$ be a linear transformation with matrix A_{CD} with respect to bases C and D. Then the matrix A_{BD} for the composite linear transformation $L_2 \circ L_1 : V_1 \to V_3$ with respect to bases B and D is the product $A_{CD}A_{BC}$.*

Example 13. Let L_1 be the linear operator on $\mathbb{R}^2$ representing the counter-clockwise rotation of vectors in $\mathbb{R}^2$ through a fixed angle θ, and let L_2 be the linear operator on $\mathbb{R}^2$ representing the reflection of vectors in $\mathbb{R}^2$ through the $x - axis$. If A_1 and A_2 denote, respectively, the matrices for L_1 and L_2 with respect to the standard basis for $\mathbb{R}^2$, then we know that

$$A_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \textit{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

By Theorem 8, the matrix A for the composite linear operator $T_{@} \circ T_1$ with respect to the standard basis for $\mathbb{R}^2$ is given by

$$A = A_2A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{bmatrix}.$$

Exercise 11 Let $T : \mathbb{R}^2 \to R^2$ be the linear operator given by $T([x, y]) = [3x-4y, -x+2y]$. Find th matrix for T with respect to the basis $\{[4, -3], [3, -2]\}$, using the method of similarity.