Weekly e-material (7/4/2020 to 10/4/2020) (100)

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(8) Use the Gram - Schmidt process to find an orthogonal basis for the subspaces of $\mathbb{R}^4$ spanned by the set $\{[2,1,0,-1], [1,1,-1],$ $[1,-2,1,1]\}$.

$\to$ Let $B = \{[2,1,0,-1], [1,1,1,-1], [1,-2,1,1]\}$.

be a subset of $\mathbb{R}^4$.

Let $W = \text{Span}\{B\}$ be a subspace of $\mathbb{R}^4$.

Using the independence test method, we have to show that B is linearly independent.

$$[A|0] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + 2R_3]{R_2 \to R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, B is linearly independent.

Now, we will apply the Gram-Schmidt process to find an orthogonal basis for W. Let

$\vec{w}_1 = [2,1,0,-1]$, $\vec{w}_2 = [1,1,1,-1]$, $\vec{w}_3 = [1,-2,1,1]$.

Following the Gram-Schmidt process, we get.

$$\vec{v}_1 = \vec{w}_1 = [2,1,0,-1].$$

$$\vec{v}_2 = \vec{w}_2 - \frac{(\vec{w}_2 \cdot \vec{w}_1)\,\vec{w}_1}{\|\vec{w}_1\|^2}$$

$$= [1,1,1,-1] - \frac{[1,1,1,-1][2,1,0,-1]}{(\sqrt{4+1+1})^2}[2,1,0,-1]$$

$$= [1,1,1,-1] - \frac{2+1+0+1}{6}[2,1,0,-1]$$

$$= [1,1,1,-1] - \frac{4}{6}[2,1,0,-1]$$

$$= \left[1-\frac{4}{3}, 1-\frac{2}{3}, 1, -1+\frac{2}{3}\right]$$

$$= \left[\frac{-1}{3}, \frac{1}{3}, 1, \frac{-1}{3}\right]$$

$$= \frac{1}{3}[-1, 1, 3, -1]$$

Multiplying this vector by 3 to avoid fractions, we get $\vec{v}_2 = [-1,1,3,-1]$. Notice that $\vec{v}_2$ is orthogonal to $\vec{v}_1$. Finally,

$$\vec{v_3} = \vec{w_3} - \frac{\vec{w_3}\cdot\vec{w_1}}{\|\vec{w_1}\|^2}\vec{w_1} - \frac{\vec{w_3}\cdot\vec{w_2}}{\|\vec{v_2}\|^2}\vec{v_2}.$$

$$= [1,-2,1,1] - \frac{[1,-2,1,1][2,1,0,-1]}{6}[2,1,0,-1] - \frac{[1,-2,1,1][-1,1,3,-1]}{12}[-1,1,3,-1]$$

$$= [1,-2,1,1] - \frac{2-2-1}{6}[2,1,0,-1] - \frac{-1-2+3-1}{12}[-1,1,3,-1]$$

$$= [1,-2,1,1] + \frac{1}{6}[2,1,0,-1] + \frac{1}{12}[-1,1,3,-1]$$

$$= \frac{1}{12}\{[12,-24,12,12] + [4,2,0,-2] + [-1,1,3,-1]\}$$

$$= \frac{1}{12}\{[12+4-1, -24+2+1, 12+3, 12-2-1]\}$$

$$= \frac{1}{12}[15, -21, 15, 9]$$

$$= \frac{3}{12}[5, -7, 5, 3]$$

$$= \frac{1}{4}[5, -7, 5, 3]$$

Multiplying this vector by 4 to avoid fractions, we get $\vec{v_3} = [5,-7,5,3]$. Notice that $\vec{v_3}$ is orthogonal to both $\vec{v_1}$ and $\vec{v_2}$.

Hence the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \{[2,1,0,-1], [-1,1,3,-1], [5,-7,5,3]\}$ is an orthogonal basis for W

properties of orthogonal Complements.

Theorem :- Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$, and $W \cap W^\perp = \{\vec{0}\}$.

proof:- Since $\vec{0}.\vec{u} = \vec{0} \; \forall \; \vec{u} \in W$. Therefore

$\vec{0} \in W^\perp$. Hence $W^\perp \neq \{\phi\}$.

Thus, in order to show that $W^\perp$ is a subspace of $\mathbb{R}^n$, we must show that $W^\perp$ is closed under addition and scalar multiplication.

Let $\vec{x}$ and $\vec{y}$ be any two vectors in $W^\perp$. Then,

for all $\vec{v} \in W$, we have

$$(\vec{x}+\vec{y}).\vec{v} = \vec{x}.\vec{v} + \vec{y}.\vec{v}$$

$$= 0+0 \quad [\because \vec{x}.\vec{v} = 0 \; \& \; \vec{y}.\vec{v} = 0, \text{ for all } \vec{v} \in W]$$

$$= 0$$

Therefore, $\vec{x}+\vec{y} \in W^\perp$.

Let $\vec{x}$ be any vector in $W^\perp$ and $\alpha \in \mathbb{R}$. Then,

for all $\vec{v} \in W$, we have

$$(\alpha\vec{x}).\vec{v} = \alpha(\vec{x}.\vec{v})$$

$$= \alpha.0$$

$$= 0.$$

Therefore, $\alpha \vec{x} \in W^{\perp}$.

Hence, $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

finally, to show that $W \cap W^{\perp} = \{\vec{0}\}$.

Let $\vec{w} \in W \cap W^{\perp}$. Then $\vec{w} \in W$ and $\vec{w} \in W^{\perp}$, and hence $\vec{w} \cdot \vec{w} = 0$

$$\Rightarrow \|\vec{w}\|^2 = 0$$
$$\Rightarrow \|\vec{w}\| = 0$$
$$\Rightarrow \vec{w} = \vec{0}.$$

Therefore, $W \cap W^{\perp} = \{\vec{0}\}$.

Theorem :- Let W be a subspace of $\mathbb{R}^n$. Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be an orthogonal basis for W contained in an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$. Then $\{\vec{v}_{k+1}, \vec{v}_{k+2}, \dots, \vec{v}_n\}$ is an orthogonal basis for $W^{\perp}$.

Corollarly :- Let W be a subspace of $\mathbb{R}^n$. Then

$$\dim(W) + \dim(W^{\perp}) = \dim(\mathbb{R}^n) = n.$$

proof :- Let W be a subspace of $\mathbb{R}^n$ of dimension k. By Theorem (Let $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ obtained by applying the Gram-

Schmidt process to $\mathcal{B}$ is an orthogonal basis for W).

We have, W has an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$.

We thus have an orthogonal set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ of non-zero vectors in $\mathbb{R}^n$. Hence by theorem (Let W be a subspace of $\mathbb{R}^n$. Then any orthogonal set of non-zero vectors in W can be enlarged to an orthogonal basis for W), it can be enlarged to an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$. Then by previous theorem, $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $W^\perp$, and so $\dim W^\perp = n-k$. Hence

$$\dim(W) + \dim(W^\perp) = n = \dim(\mathbb{R}^n).$$

Example 1) For each of the following subspaces W of $\mathbb{R}^n$, find a basis for $W^\perp$, and verify that $\dim(W) + \dim(W^\perp) = n = \dim(\mathbb{R}^n)$

(a) In $\mathbb{R}^2$, $W = \text{span}\{[4,-3]\}$

(b) In $\mathbb{R}^3$, $W = \text{span}\{[-1, 3, 2]\}$

(c) In W = the plane $x + 4y - 2z = 0$.

→ (a) $W = \text{span}\{[4,-3]\} = \{\alpha[4,-3] \mid \alpha \in \mathbb{R}\}$ 107

If $\vec{v} = [x,y] \in W^{\perp}$ then

$\vec{v}\cdot\vec{w} = 0 \;\forall\; \vec{w} \in W$. we have,

$\vec{v}\cdot[4,-3] = [x,y]\cdot[4,-3] = 0$

$\Rightarrow 4x - 3y = 0$

$\Rightarrow x = \frac{3}{4}y$

Take $y = 4$ then $x = 3$.

$\vec{v} = [3,4]$.

Therefore, $W^{\perp} = \text{span}\{[3,4]\}$

$= \{\alpha[3,4] \mid \alpha \in \mathbb{R}\}$

Hence $\dim W + \dim W^{\perp} = 1 + 1 = 2 = \dim(\mathbb{R}^2)$

(b) Let $W = \text{span}\{[-1,3,2]\}$ in $\mathbb{R}^3$.

If $\vec{v} = [x,y,z] \in W^{\perp}$ then

$\vec{v}\cdot[-1,3,2] = [x,y,z]\cdot[-1,3,2] = 0$

$\Rightarrow -x + 3y + 2z = 0$

First, take $z = 1$ and $y = 0$.

$\Rightarrow x = 2$.

& take $z = 0$ and $y = 1$, we get

$x = 3$.

Therefore,

$$W^{\perp} = \text{span}\{[2,0,1],[3,1,0]\}.$$

$$= \{a[2,0,1]+b[3,1,0] \mid a,b \in \mathbb{R}\}$$

$$= \{[2a+3b,b,a] \mid a,b \in \mathbb{R}\}.$$

$$\dim(W^{\perp}) + \dim(W) = 2+1 = 3 = \dim(\mathbb{R}^3).$$

(c) $$W = \{[x,y,z] \in \mathbb{R}^3 \mid x+4y-2z=0\}$$

$$= \{[-4y+2z,y,z] \mid y,z \in \mathbb{R}\}.$$

$$= \{[-4y,y,0]+[2z,0,z] \mid y,z \in \mathbb{R}\}$$

$$= \{y[-4,1,0]+z[2,0,1] \mid y,z \in \mathbb{R}\}.$$

$$= \text{span}\{[-4,1,0],[2,0,1]\}.$$

Since $\{[-4,1,0], [2,0,1]\}$ is L.I.

Therefore, $\dim(W) = 2$.

$$W^{\perp} = \{[x,y,z] \in \mathbb{R}^3 \mid [x,y,z]\cdot[-4,1,0]=0 \text{ and } [x,y,z]\cdot[2,0,1]=0\}.$$

$$= \{[x,y,z] \in \mathbb{R}^3 \mid -4x+y=0 \text{ and } 2x+z=0\}.$$

$$= \text{span}\{[-\tfrac{1}{2},-2,1]\}.$$

Therefore, $\dim(W^{\perp}) = 1$. Now, we have

$$\dim(W) + \dim(W^{\perp}) = 2+1 = 3 = \dim(\mathbb{R}^3)$$

Corollary :- Let W be a subspace of $\mathbb{R}^n$. Then

$$(W^{\perp})^{\perp} = W.$$

Let $\vec{x} \in W$.

$$\vec{x}.\vec{y} = 0 \text{ for all } \vec{y} \in W^{\perp}.$$

$$\vec{x} \in (W^{\perp})^{\perp}.$$

$$\Rightarrow W \subseteq (W^{\perp})^{\perp}.$$

In order to show that $W = (W^{\perp})^{\perp}$, it is enough to show that $\dim W = \dim(W^{\perp})^{\perp}$.

By above Corollary, we have.

$$\dim(W^{\perp}) + \dim((W^{\perp})^{\perp}) = n = \dim(\mathbb{R}^n)$$

& $\dim(W^{\perp}) + \dim(W) = n = \dim(\mathbb{R}^n)$.

~~dim~~.

we have,

$$\dim((W^{\perp})^{\perp}) = n - \dim(W^{\perp})$$

$$= \dim(W).$$

$$\therefore \dim((W^{\perp})^{\perp}) = \dim(W).$$

Hence $(W^{\perp})^{\perp} = W$.

Example :- For the subspace $W = \{[x,y,z] \mid 2x-3y+4z=0\}$ find the orthogonal complement $W^{\perp}$ and verify that $\dim(W) + \dim(W^{\perp}) = \dim(\mathbb{R}^n)$.

$\rightarrow$ $W = \{[x,y,z] \mid 2x-3y+4z=0\}$

$= \{[x,y,z] \mid [x,y,z][2,-3,4] = 0\}$

Let $Y = \text{span}\{[2,-3,4]\}$. Then

$Y^{\perp} = W$ $[\because \text{for } \vec{v} \in W,\ \vec{v}.\vec{w} = 0\ \forall \vec{w} \in Y]$

$\Rightarrow W^{\perp} = (Y^{\perp})^{\perp} = Y$.

Clearly, $\dim(W^{\perp}) = 1$

$\dim W = \dim\{[\frac{3y+4z}{2}, y, z] \mid y,z \in \mathbb{R}\}$

$= \dim\{y[\frac{3}{2},1,0] + z[-\frac{4}{2},0,1] \mid y,z \in \mathbb{R}\}$

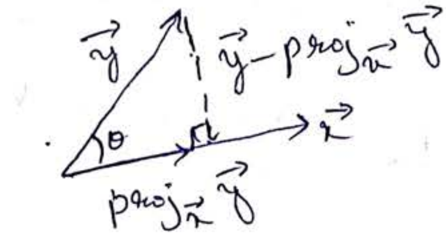
$= \dim\{y[3,2,0] + z[-2,0,1] \mid y,z \in \mathbb{R}\}$

$= 2$ $[\because [3,2,0]$ & $[-2,0,1]$ are L.I. each other$]$

Therefore, $\dim(W) + \dim(W^{\perp}) = 2+1 = 3 = \dim(\mathbb{R}^3)$

Orthogonal projections.

Recall that if $\vec{x}$ is non-zero vector in $\mathbb{R}^2$, then every vector $\vec{y}$ in $\mathbb{R}^2$ can be decomposed as the sum of two component vectors, $proj_{\vec{x}}\vec{y}$ and $\vec{y} - proj_{\vec{x}}\vec{y}$, where $proj_{\vec{x}}\vec{y}$ is the projection vector of $\vec{y}$ onto $\vec{x}$ which is parallel to $\vec{x}$ and the second is orthogonal to $\vec{x}$.



Theorem (projection theorem)

Let W be a subspace of $\mathbb{R}^n$. Then every vector $\vec{v}$ $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{v} = \vec{w}_1 + \vec{w}_2 \text{ ——— (1)}$$

where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$. In fact if $\{\vec{u}_1, \dots \vec{u}_k\}$ is any orthonormal basis for W, then

$$\vec{w}_1 = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot u_2)\vec{u}_2 + \cdots + (\vec{v}\cdot\vec{u}_k)\vec{u}_k \text{ —(2}$$

and $\vec{w}_2 = \vec{v} - \vec{w}_1$

The vector $\vec{w}_1$ in (1) is called the orthogonal projection of $\vec{v}$ onto W and is written as $proj_{W}\vec{v}$.

Definition (orthogonal projection onto a subspace)

Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$, and let $\vec{v} \in \mathbb{R}^n$. Then the orthogonal projection of $\vec{v}$ onto W is defined to the vector

$$\text{proj}_W \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (v\cdot\vec{u}_2)\vec{u}_2 + \cdots + (v\cdot u_k)\vec{u}_k.$$

If W is the trivial subspace of $\mathbb{R}^n$, then

$$\text{proj}_W \vec{v} = \vec{0}.$$

Projection theorem:- Let W be a subspace of $\mathbb{R}^n$.

Then every vector $\vec{v}$ in $\mathbb{R}^n$ can be expressed in a unique way as $\vec{w}_1 + \vec{w}_2$, where $\vec{w}_1 = \text{proj}_W \vec{v} \in W$ and $\vec{w}_2 = \vec{v} - \text{proj}_W \vec{v} \in W^\perp$. Moreover, $\vec{w}_2$ can also be expressed as $\text{proj}_{W^\perp} v$.

Example 1) Find the orthogonal projection of $\vec{v} = [2, 1, -3]$ onto the subspace W of $\mathbb{R}^3$ spanned by the orthogonal vectors $\vec{v}_1 = [1, 0, 1]$ and $\vec{v}_2 = [0, 1, -1]$.

$\rightarrow$

The subspace W of $\mathbb{R}^3$ is defined to be

$$W = \text{span}\{[1,0,1], [0,1,-1]\}$$

$$= \text{span}\left\{\frac{1}{\sqrt{2}}[1,0,1], \frac{1}{\sqrt{2}}[0,1,-1]\right\},$$ where

we have normalized the vectors $[1,0,1]$ and $[0,1,-1]$ to get an orthonormal basis for W.

Hence, by definition of $\text{proj}_W \vec{v}$, we have

$$\text{proj}_W \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2,$$

where $\vec{u}_1 = \frac{1}{\sqrt{2}}[1,0,1]$ and $\vec{u}_2 = \frac{1}{\sqrt{2}}[0,1,-1]$.

$$\therefore \text{proj}_W \vec{v} = \left([2,1,-3]\cdot\frac{1}{\sqrt{2}}[1,0,1]\right)\frac{1}{\sqrt{2}}[1,0,1] +$$

$$\left([2,1,-3]\frac{1}{\sqrt{2}}[0,1,-1]\right)\frac{1}{\sqrt{2}}[0,1,-1].$$

$$= \left(\frac{1}{\sqrt{2}}(2+0-3)\right)\frac{1}{\sqrt{2}}[1,0,1] +$$

$$\left(\frac{1}{\sqrt{2}}(0+1+3)\right)\frac{1}{\sqrt{2}}[0,1,-1]$$

$$= \frac{1}{2}[-1,0,-1] + \frac{1}{2}\cdot 4\,[0,1,-1]$$

$$= \left[-\frac{1}{2}, 0, -\frac{1}{2}\right] + [0,2,-2]$$

$$= \left[-\frac{1}{2}, 2, -\frac{5}{2}\right]$$

$$= \frac{1}{2}[-1,4,-5]$$

(2) Consider the subspace $W = \text{span}\{[2,1,-1]\}$ of $\mathbb{R}^3$. Let $\vec{v} = [1,4,-2]$. Find $\text{proj}_W \vec{v}$ and decompose it into $\vec{w}_1 + \vec{w}_2$, where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$. Is the decomposition unique?

$\rightarrow$ $W = \text{span}\{[2,1,-1]\}$

$= \text{span}\{\frac{1}{\sqrt{6}}[2,1,-1]\}$

$$\vec{w}_1 = \left(\vec{v} \cdot \frac{1}{\sqrt{6}}[2,1,-1]\right) \frac{1}{\sqrt{6}}[2,1,-1]$$

$$= \left(\frac{1}{\sqrt{6}}[1,4,-2]\cdot[2,1,-1]\right) \frac{1}{\sqrt{6}}[2,1,-1]$$

$$= \frac{1}{\sqrt{6}}(2+4+2)\frac{1}{\sqrt{6}}[2,1,-1] = \frac{1}{6}\, 8\, [2,1,-1]$$

$$= \frac{1}{3}[8,4,-4]$$

$$\vec{w}_2 = \vec{v} - \vec{w}_1 = [1,4,-2] - \frac{1}{3}[8,4,-4]$$

$$= \frac{1}{3}[3-8, 12-4, -6+4]$$

$$= \frac{1}{3}[-5,8,-2]$$

Clearly, $\vec{w}_2 \cdot \frac{1}{\sqrt{6}}[2,1,-1] = \frac{1}{3}[-5,8,-2]\frac{1}{\sqrt{6}}[2,1,-1]$

$$= \frac{1}{3\sqrt{6}}(-10+8+2) = 0.$$

$\therefore \vec{w}_2 \in W^\perp$ and $\vec{v} = \vec{w}_1 + \vec{w}_2$.

(2) Let W be the subspace of $\mathbb{R}^3$ whose vectors (beginning at the origin) lie in the plane $2x+y+z=0$, let $\vec{v} = [-6, 10, 5]$. Find $\text{proj}_W \vec{v}$ and decompose $\vec{v}$ into $\vec{w_1}+\vec{w_2}$, where $w_1 \in W$ and $w_2 \in W^\perp$.

$\rightarrow$ $W = \{[x,y,z] \in \mathbb{R}^3 / 2x+y+z=0\}$.

By substituting first $x=1, y=0$ and then $x=0, y=1$ in the equation $2x+y+z=0$ and solving z, we find that $[1,0,-2]$ and $[0,1,-1]$ are two linearly independent vector in W. Let $\vec{x_1} = [1,0,-2]$ and $\vec{x_2} = [0,1,-1]$. We now use Gram-Schmidth process on the basis $\{\vec{x_1}, \vec{x_2}\}$ to obtained the orthogonal basis for W. Following Gram-Schmidth process, we get

$$\vec{v_1} = \vec{x_1} = [1,0,-2].$$

$$\vec{v_2} = \vec{x_2} - \left(\frac{\vec{x_2}\cdot\vec{x_1}}{\|\vec{x_1}\|^2}\right)\vec{x_1}$$

$$= [0,1,-1] - \frac{[0,1,-1][1,0,-2]}{5}[1,0,-2]$$

$$= [0,1,-1] - \frac{2}{5}[1,0,-2]$$

$$= \left[-\frac{2}{5}, 1, -\frac{1}{5}\right].$$

To avoid fractions, we replace this vector with an appropriate scalar multiple. Multiplying by 5, we get $\vec{v_2} = [-2,5,-1]$. Notice that $\vec{v_2}$ is orthogonal to $\vec{v_1}$. Hence, $\{\vec{v_1}, \vec{v_2}\} = \{[1,0,-2], [-2,5,-1]\}$

is an orthogonal basis for W. Normalizing $\vec{v}_1$ and $\vec{v}_2$, we obtain.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{5}}[1,0,-2] \text{ and } \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{30}}[-2,5,-1]$$

Hence $\{\vec{u}_1 = \frac{1}{\sqrt{5}}[1,0,-2],\ \vec{u}_2 = \frac{1}{\sqrt{30}}[-2,5,-1]\}$ is an orthonormal basis for W. So, by definition

$$\vec{w}_1 = \text{proj}_W \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2$$

$$= \left([-16,10,5]\cdot\left(\frac{1}{\sqrt{5}}[1,0,-2]\right)\right)\left(\frac{1}{\sqrt{5}}[1,0,-2]\right)$$

$$+\left([-6,10,5]\cdot\frac{1}{\sqrt{30}}[-2,5,-1]\right)\left(\frac{1}{\sqrt{30}}[-2,5,-1]\right)$$

$$= \left(-\frac{16}{5}\right)[1,0,-2] + \left(\frac{57}{30}\right)[-2,5,-1]$$

$$= \left[-\frac{16}{5}, 0, \frac{32}{5}\right] + \left[-\frac{114}{30}, \frac{285}{30}, -\frac{57}{30}\right]$$

$$= \left[-7, \frac{19}{2}, \frac{9}{2}\right]$$

Notice that $\vec{w}_1 \in W$. finally,

$$\vec{w}_2 = \vec{v} - \vec{w}_1$$

$$= [-6,10,5] - \left[-7, \frac{19}{2}, \frac{9}{2}\right]$$

$$= \left[1, \frac{1}{2}, \frac{1}{2}\right]$$

$$\therefore \vec{w}_2 = \left[1, \frac{1}{2}, \frac{1}{2}\right]$$

clearly, $\vec{w}_2 \in W^{\perp}$ ($\because \vec{w}_2\cdot\vec{u}_1 = 0$ and $\vec{w}_2\cdot\vec{u}_2 = 0$). Hence

Finally, $\vec{w}_1 = \text{proj}_W \vec{v} = \vec{v} - \text{proj}_{W^\perp} \vec{v}$

$$= [-6, 10, 5] - [1, \tfrac{1}{2}, \tfrac{1}{2}]$$

$$= \tfrac{1}{2} [-7, 10 - \tfrac{1}{2}, 5 - \tfrac{1}{2}]$$

$$= [-7, \tfrac{19}{2}, \tfrac{9}{2}]$$

Thus, we have decomposed $\vec{v} = [-6, 10, 5]$ as

$$\vec{v} = \vec{w}_1 + \vec{w}_2 = [-7, \tfrac{19}{2}, \tfrac{9}{2}] + [1, \tfrac{1}{2}, \tfrac{1}{2}],$$

$$= \cancel{[-6, 10, 5]},$$

where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$.

We have decomposed $\vec{v} = [-6, 10, 5]$ as

$$\vec{v} = \vec{w_1} + \vec{w_2} = \left[-7, \frac{19}{2}, \frac{9}{2}\right] + \left[1, \frac{1}{2}, \frac{1}{2}\right].$$

where $\vec{w_1} \in W$ and $\vec{w_2} \in W^{\perp}$.

Alternative Method :- $W = \{[x, y, z] \in \mathbb{R}^3 / 2x + y + z = 0\}$

$$= \{[x, y, z] \in \mathbb{R}^3 / [x, y, z] \cdot [2, 1, 1] = 0\}$$

That is, W is the set of all vectors orthogonal to $[2, -2, 1]$, and hence orthogonal to the subspace $Y = \text{span}\{[2, 1, 1]\}$. Thus, by the definition of orthogonal Complement, $W = Y^{\perp}$. This implies

$$W^{\perp} = (Y^{\perp})^{\perp} = Y = \text{span}\{[2, 1, 1]\}$$

$$= \text{span}\left\{\frac{1}{\sqrt{6}}[2, 1, 1]\right\},$$

Where we have normalized the vector $[2, 1, 1]$ to get an orthonormal basis for Y. Thus, by the definition of $\text{proj}_{W^{\perp}}\vec{v} = \text{proj}_{Y}\vec{v} = \vec{w_2}$.

$$\vec{w_2} = \left([-6, 10, 5] \cdot \left[\frac{1}{\sqrt{6}}[2, 1, 1]\right]\right) \frac{1}{\sqrt{6}}[2, 1, 1]$$

$$= \frac{1}{6}(-12 + 10 + 5)[2, 1, 1].$$

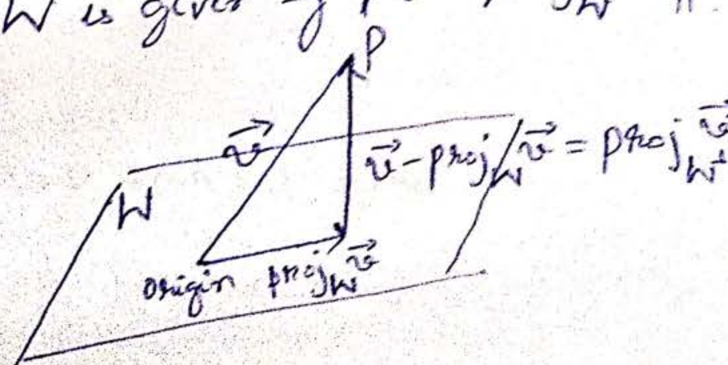
$$= \frac{1}{2}[2, 1, 1] = \left[1, \frac{1}{2}, \frac{1}{2}\right].$$

Distance from of a point to a subspace: An application.

Definition (Minimum Distance)

Let W be a subspace of $\mathbb{R}^n$, and assume all vectors in W having initial point at the origin. Let p be any point in n-dimensional space. Then the minimum distance from p to W is defined to be the shortest distance between p and the terminal point of any vector in W.

Theorem :- Let W be a subspace of $\mathbb{R}^n$, and let p be a point in n-dimensional space. If $\vec{v}$ is the vector from the origin to p, then the minimum distance from p to W is given by $\|\vec{v} - proj_W \vec{v}\|$.



Example :- Let W be the subspace of $\mathbb{R}^3$ whose vectors lie in the plane $2x-y+2z=0$. Find the minimum distance from the point $P(4,1,-3)$ to W.

$\rightarrow$

$$W = \{[x,y,z] \in \mathbb{R}^3 \,/\, 2x-y+2z=0\}$$

$$= \{[x,y,z] \in \mathbb{R}^3 \,/\, [x,y,z][2,-1,2]=0\}$$

Let $Y = \text{span}\{[2,-1,2]\}$. Then

$Y^{\perp} = W$. This implies $W^{\perp} = (Y^{\perp})^{\perp}$
$= Y$.

$$\Rightarrow W^{\perp} = Y = \text{span}\{[2,-1,2]\}$$

$$= \text{span}\left\{\frac{1}{\sqrt{9}}[2,-1,2]\right\}$$

$$= \text{span}\left\{\frac{1}{3}[2,-1,2]\right\}.$$

Where we have normalized $[2,-1,2]$ to get a orthonormal basis for Y. Hence, by definition of $\text{proj}_{W^{\perp}}\vec{v}$.

$$\vec{v} - \text{proj}_W \vec{v} = \text{proj}_{W^{\perp}} \vec{v}$$

$$= \left([4,1,-3]\frac{[2,-1,2]}{3}\right)\left(\frac{1}{3}[2,-1,2]\right)$$

$$[2,-1,2]$$

$$= \frac{1}{9}(8-1-6)$$

$$= \frac{1}{9}[2,-1,2].$$

Hence, the minimum distance from $P(4,1,-3)$ to G

$$\|\vec{v} - \text{proj}_W \vec{v}\| = \sqrt{\left(\frac{1}{9}\right)^2(4+1+4)} = \frac{1}{9}3 = \frac{1}{3}.$$