Weekly e-material (30/3/2020 to 3/4/2020)
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One-to-one linear transformations and linear independence.

<u>Theorem 1</u> :- Let V and W be vector spaces, and let $T: V \longrightarrow W$ be a one-to-one linear transformation. If S is a linear independent subset of V, then T(S) is a linear independent in W.

proof :- Let S be a linear independent subset of V. To show that T(S) is a linear independent subset of W. Suppose $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a finite subset of T(S), where $v_1, v_2, \dots, v_n \in S$ and suppose there are scalars $a_1, a_2, \dots a_n \in \mathbb{R}$, such that

$$a_1 T(v_1) + a_2 T(v_2) + \dots + a_n T(v_n) = 0_W$$

$$\Rightarrow T(a_1 v_1 + a_2 v_2 + \dots + a_n v_n) = 0_W$$

$$\Rightarrow a_1 v_1 + a_2 v_2 + \dots + a_n v_n \in \text{Ker}\, T = \{0_V\}$$
($\because$ T is one-to-one)

$\Rightarrow \quad a_1 v_1 + a_2 v_2 + \dots + a_r v_r = 0_V$. (6 %)

Since S is L.I., therefore

$$a_1 = a_2 = \dots = a_r = 0.$$

Therefore $\{T(v_1), T(v_2), \dots, T(v_r)\}$ is linearly independent.

Hence $T(S)$ is linearly independent in W.

Spanning sets and onto linear transformations

<u>Theorem 2</u> :- Let V and W be vector spaces, and let $T: V \to W$ be an onto linear transformation. If any subset S of V spans V, then $T(S)$ spans W.

Proof :- Suppose that S spans V. To prove that $T(S)$ spans W, We must show that any vector $w \in W$ can be expressed as a linear combination of vectors in $T(S)$. Since T is onto, there is a vector $v \in V$ such that $T(v) = w$. Since S spans V, there exist vectors $v_1, v_2, \dots, v_r \in V$ and scalars $a_1, a_2, \dots, a_r \in \mathbb{R}$ such that

$v = a_1 v_1 + a_2 v_2 + \dots + a_r v_r$. Then

$$w = T(v) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$$

$$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$$

$$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$$

$$\Rightarrow w \in \operatorname{Span}(T(S))$$

Therefore $W \subseteq \operatorname{span}(T(S))$. Hence

$$W = \operatorname{Span}(T(S)).$$

Isomorphism.

<u>Definition</u> :- Let V and W be vector spaces and let $T : V \to W$ be a linear transformation.

T is said to be an invertible linear transformation if there is a function $S : W \to V$ such that $(S \circ T)(v) = v$ for all $v \in$ and $(T \circ S)(w) = w$, for all $w \in W$. Such a function S is called an inverse of T.

Remark :- If a linear transformation $T : V \to W$ is invertible, its inverse is unique and is denoted by T^{-1}.

Definition :- Let V and W be vector spaces. A linear Transformation $T: V \to W$ that is both one-to-one and onto is called an isomorphism of V onto W.

Example 1) Show that the mapping $T: M_{mn} \to M_{nm}$ defined by $T(A) = A^T$ is an isomorphism.

$$\rightarrow \quad T(A+B) = (A+B)^T = A^T + B^T$$
$$= T(A) + T(B),$$
for all $A, B \in M_{mn}$

$$T(\alpha A) = (\alpha A)^T = \alpha A^T = \alpha T(A) \text{ for all }$$
$A \in M_{mn}$ & $\alpha \in \mathbb{R}$.

Therefore, T is a linear Transformation.

Now, we have to show that T is one-to-one and onto.

Let $A \in \operatorname{Ker} T$, then $T(A) = O$, O is zero $n \times m$ matrix.

$$\Rightarrow A^T = O \Rightarrow a_{ji} = 0 \ \forall \ j \text{ and } i,$$
where $A = [a_{ij}]$.

Therefore, all entries of A are zero, hence A is a zero $m \times n$-matrix.

Thus $\operatorname{Ker} T \subseteq \{O\}$.

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This implies $\ker T = \{0\}$.

Hence T is one-to-one.

Let $B \in M_{mn}$ be any element.

Then B^T is $m \times n$-matrix, since B is $n \times m$-matrix. Therefore $B^T \in M_{mn}$ and we have $T(B^T) = (B^T)^T = B$.

~~Therefore~~ Hence T is onto.

Hence T is an isomorphism.

(2) Show that the mapping $T: P_2 \to P_2$ given by $T(p(x)) = p(x) + p'(x)$ is an isomorphism, where P_2 is the vector space of all polynomials of degree ≤ 2.

→ First we show that T is a linear operator.

$$T(p(x) + q(x)) = (p(x) + q(x)) + (p(x) + q(x))'$$
$$= p(x) + q(x) + p'(x) + q'(x)$$
$$= (p(x) + p'(x)) + (q(x) + q'(x))$$
$$= T(p(x)) + T(q(x)).$$

& $T(\alpha\, p(x)) = (\alpha\, p(x)) + (\alpha\, p(x))'$ (72)

$= \alpha\, p(x) + \alpha\, p'(x).$

$= \alpha\,(p(x) + p'(x))$

$= \alpha\, T(p(x))$

$\therefore$ T is a linear operator.

Now, we have to show that T is one-to-one and onto.

Let $p(x) \in \ker T$. Then $T(p(x)) = 0$

$\Rightarrow p(x) + p'(x) = 0$

Suppose $p(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$.

Then, $(ax^2 + bx + c) + (2ax + b) = 0$

$\Rightarrow ax^2 + (b + 2a)x + (c + b) = 0$

$\Rightarrow a = 0,\ b + 2a = 0\ \&\ c + b = 0$

$\Rightarrow a = b = c = 0.$

Therefore, $p(x) = 0.$

$\Rightarrow \ker T \subseteq \{0\}.$

$\Rightarrow \ker T = \{0\}.$

Therefore, T is one-to-one.

2 Let $q(x) = a x^2 + bx + c$ be any element in P_2. Define $p(x) = ax^2 + (b-2a)x +$ $(c+2a-b)$ ~~Bx~~. Then

$$T(p(x)) = p(x) + p'(x)$$

$$= ax^2 + (b-2a)x + (c+2a-b) + 2ax$$

$(b-2a)$

$$= ax^2 + bx + c = q(x).$$

Therefore, T is onto.

Hence T is an isomorphism.

Theorem :- Let $T : V \to W$ be a linear transformation Then T is an isomorphism if and only if T is an invertible linear transformation. Moreover, if T is invertible, then T^{-1} is also a linear transformation.

proof :- First, suppose that $T : V \to W$ is an invertible linear transformation, and $T^{-1} : W \to V$ is its inverse. Then $T^{-1}(T(v)) = v$, for all $v \in V$ and $T(T^{-1}(w)) = w$; for all $w \in W$.

We will show that T is one-to-one, i.e. it is equivalent to show that $\ker T = \{0_V\}$.

Let $v \in \ker T$. Then $T(v) = 0_W$.

$\Rightarrow T(v) = T(0_V)$ $[\because T(0_V) = 0_W]$

$\Rightarrow T^{-1}(T(v)) = T^{-1}(T(0_V))$

$\Rightarrow v = 0_V$ $[\because T^{-1}(T(v)) = v]$

$\Rightarrow \ker T \subseteq \{0_V\}$.

Therefore, $\ker T = \{0_V\}$.

Hence T is one-to-one.

Let $w \in W$ be any element. Then

$T(T^{-1}(w)) = w$.

Since $T^{-1}(w) \in V$, therefore T is onto.

Hence T is an isomorphism.

Conversely, suppose T is an isomorphism.

We have to show that T is invertible.

Since T is onto, for each $w \in W$, there exist $v \in V$ such that $T(v) = w$.

Since T is one-to-one, v is the unique vector in V such that $T(v) = w$.

(75)

Now, defined the mapping $S : W \to V$ defined by $S(w) = v$, where v is the unique vector in V such that $T(v) = w$. The mapping S is obviously well-defined and has domain W. Also, $(T \circ S)(w) = T(S(w)) = T(v) = w$.

& $S \circ T(v) = S(T(v)) = v$ [Since v is the unique pre-image of $T(v)$ under T].

Thus, T is invertible and $S = T^{-1}$.

Finally, we have to show that S is a linear transformation.

Let $w_1, w_2 \in W$ and $\alpha \in \mathbb{R}$.

$$T(S(w_1 + w_2)) = w_1 + w_2$$

$$= T(S(w_1)) + T(S(w_2))$$

$$= T(S(w_1) + S(w_2))$$

$\Rightarrow T(S(w_1 + w_2))$

$\therefore T$ is one-to-one.

$\therefore S(w_1 + w_2) = S(w_1) + S(w_2)$.

& $T(S(\alpha w)) = \alpha w$

$$= \alpha T(S(w))$$

$$= T(\alpha(S(w)))$$

$\Rightarrow S(\alpha w) = \alpha S(w)$.

Thus, $S = T^{-1}$ is a linear transformation.

Example (1): Let θ be a fixed angle in $\mathbb{R}^2$ and

$T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation linear operator defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker T$.

Then $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \cos\theta\, x - \sin\theta\, y \\ \sin\theta\, x + \cos\theta\, y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow \cos\theta\, x - \sin\theta\, y = 0 \quad -(1)$

$\sin\theta\, x + \cos\theta\, y = 0 \quad -(2)$

$\Rightarrow \cos^2\theta\, x - \cos\theta\sin\theta\, y = 0$ $\left[\because \cos\theta \times (1) \right.$

& $\sin^2\theta\, x - \sin\theta\cos\theta\, y = 0$ $\left. \& \sin\theta \times (2)\right]$

Adding above two eq^{n}s., we have

$\cos^2\theta\, x + \sin^2\theta\, x = 0$

$\Rightarrow (\cos^2\theta + \sin^2\theta)\, x = 0$ $\quad [\because \cos^2\theta + \sin^2\theta = 1]$

$\Rightarrow 1 \cdot x = 0$

$\Rightarrow x = 0$.

putting $x=0$ in eq^n. (1) & (2), we have.

$\therefore -\sin\theta\, y = 0 \Rightarrow \sin\theta\, y = 0$.

$\cancel{\Rightarrow y = 0 \ [\because \sin^2\theta + \cos^2\theta = 1]}$

& $\cos\theta\, y = 0$

$\Rightarrow \sin^2\theta\, y^2 + \sin^2\theta\, y^2 = 0$

$\Rightarrow y^2 = 0 \Rightarrow y = 0$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore $\ker T = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

$\therefore$ T is one-to-one.

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be any element in $\mathbb{R}^2$.

Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ such that

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

$\Rightarrow \cos\theta\, a - \sin\theta\, b = x$

$\Rightarrow \sin\theta\, a + \cos\theta\, b = y$.

We have to find the value of a & b.

$a = \cos\theta\, x + \sin\theta\, y$.

$b = -\sin\theta\, x + \cos\theta\, y$.

We have ~~Therefore,~~ $\begin{bmatrix} \cos\theta\, x + \sin\theta\, y \\ -\sin\theta\, x + \cos\theta\, y \end{bmatrix} \in \mathbb{R}^n$ such that

$$T\left(\begin{bmatrix} \cos\theta\, x + \sin\theta\, y \\ -\sin\theta\, x + \cos\theta\, y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, T is onto.

Hence T is an isomorphism. Thus T is invertible and T^{-1} is given by

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Theorem 3:- Let V and W both be non-trivial finite-dimensional vector spaces with ordered bases B and C, respectively, and let $T: V \to W$ be a linear transformation. Then T is an isomorphism if and only if the matrix represent A_{BC} for T with respect to B and C is non-singular.

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by $T(v) = Av$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Is T an isomorphism? Justify your answer.

$\rightarrow$ Since $|A| = 1(1-0) - 0 + 3(0) = 1$.

Then A is non-singular. Hence by Theorem 3, T is an isomorphism.

Theorem 4 :- Suppose that V, W and X are vector spaces, and suppose that $T_1 : V \to W$ and $T_2 : W \to X$ are both isomorphisms. Then the composition $T_2 \circ T_1 : V \to X$ is also an isomorphism.

Definition (Isomorphic vector spaces).

Let V and W be a vector spaces. We say that V is isomorphic to W, and write $V \cong W$, if there exists an isomorphism $T: V \to W$

Example :- Consider the linear transformation

$T: M_{22} \to P_3$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d.$$

We have to check that T is an isomorphism

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ d & h \end{bmatrix}\right) = T\left(\begin{bmatrix} a+e & b+f \\ c+d & d+h \end{bmatrix}\right)$$

$$= (a+e)x^3 + (b+f)x^2 + (c+d)x + d+h.$$

$$= (ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + dx + h)$$

$$= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right).$$

$$\& \quad T\left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right)$$

$$= (\alpha a)x^3 + (\alpha b)x^2 + (\alpha c)x + \alpha d.$$

$$= \alpha(ax^3 + bx^2 + cx + d)$$

$$= \alpha T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

$\therefore$ T is a linear transformation.

Now, we show that T is one-to-one and onto.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$ be any element.

$$\Rightarrow T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$$

$$\Rightarrow ax^3 + bx^2 + cx + d = 0 .$$

$\Rightarrow a = b = c = d = 0.$

$\Rightarrow \ker T \subseteq \{0\}.$

$\Rightarrow \ker T = \{0\}.$

Therefore, T is one-to-one.

Let $ax^3 + bx^2 + c\cdot x + d \in P_3$ be any element.

Then, $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d.$

$\therefore$ T is onto.

Hence $M_{22} \cong P_3$.

Theorem 5 :- properties of Isomorphisms

(1) Every vector space V is isomorphic to itself.

(2) If V is isomorphic to W, then W is isomorphic to V.

(3) If V is isomorphic to W, and W is isomorphic to X, then V is isomorphic to X.

Theorem 6:- Suppose $V \cong W$ and V is finite-dimensional. Then W is finite-dimensional and $\dim V = \dim W$.

proof :- Since $V \cong W$, there is an isomorphism $T: V \to W$. Let $\dim V = n$ and let $B = \{v_1, \dots, v_n\}$ be a basis for V. We claim that $T(B) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ be a basis for V.

First, we prove that $T(B)$ is L.I.

Suppose that $\cancel{c_1T(v_1)+a_2T(v_2)+\dots+a_nT(v_n)=0_W}$

$a_1T(v_1)+a_2T(v_2)+\dots+a_nT(v_n)=0_W$

$\Rightarrow a_1T(v_1)+a_2T(v_2)+\dots+a_nT(v_n)=0_W$

$\Rightarrow T(a_1v_1)+T(a_2v_2)+\dots+T(a_nv_n)=0_W$

$\Rightarrow T(a_1v_1+a_2v_2+\dots+a_nv_n)=0_W$

$\Rightarrow a_1v_1+a_2v_2+\dots+a_nv_n \in \ker T=\{0_V\}$

$\Rightarrow a_1v_1+a_2v_2+\dots+a_nv_n=0_V$.

$\Rightarrow a_1v_1+a_2v_2+\dots+a_nv_n$

Since B is L.I.

$\therefore a_1=a_2=\dots=a_n=0$.

$\therefore \{T(v_1),\dots,T(v_n)\}$ is L.I.

Next, we show that $T(B)$ spans W.

Let $w \in W$, Since T is onto,

there exist $v \in V$ such that

$T(v) = w$.

Since B is a basis for V,

there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that.

$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$.

$T(v) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$.

$\in \text{Span}\{T(v_1), \ldots, T(v_n)\}$.

$\therefore W = \text{span}\{T(B)\}$.

Hence $T(B)$ is a basis for W and $\dim W = n$. Thus $\dim V = \dim W = n$.

<u>Theorem 7</u> :- (Isomorphisms of n-dimensional vector spaces).

If V is any n-dimensional vector space, then $V \cong \mathbb{R}^n$.

proof:- Suppose that $\dim V = n$, let $B = \{v_1, \ldots v$ be an ordered basis for V. Then every vector $v \in V$ has its co-ordinatization with respect to B.

To show that V must be isomorphic to $\mathbb{R}^n$, we defined $T: V \to \mathbb{R}^n$ as follows:

$$T(v) = [v]_B.$$

$$T(v+w) = [v+w]_B$$
$$= [v]_B + [w]_B.$$
$$= T(v) + T(w)$$

& $T(\alpha v) = [\alpha v]_B = \alpha [v]_B$
$$= \alpha T(v).$$

$\therefore$ T is a linear transformation.

Let $v \in \ker(T)$.

$\Leftrightarrow T(v) = [0, 0, \ldots, 0]$

$\Leftrightarrow [v]_B = [0, 0, \ldots, 0]$

$\Leftrightarrow v = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 v_n$

$\Leftrightarrow v = 0$.

Hence $\ker T = \{0\}$ and T is one-to-one.

If $\vec{a} = [a_1, \ldots, a_n] \in \mathbb{R}^n$, then $v = a_1 v_1 + \cdots + a_n v_n$ $\in V$ such that $T(v) = [v]_B = [a_1, a_2, \ldots, a_n] = \vec{a}$. This shows that T is onto.

$\therefore$ T is an isomorphism.

Hence V is isomorphic to $\mathbb{R}^n$.

Chapter 7

Orthogonality.

Orthogonal and Orthonormal Vectors.

Definition 1) (orthogonal vectors).

A set $S = \{\vec{v}_1, \dots \vec{v}_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthogonal set of vectors if and only if every pair of distinct vectors in this set is orthogonal, that is, if and only if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$, $1 \leq i, j \leq k$.

Example 1) In $\mathbb{R}^3$; the set $\{[1, 0, -1], [1, \sqrt{2}, 1], [1, -\sqrt{2}, 1]\}$ is an orthogonal set, because.

$[1, 0, -1] \cdot [1, \sqrt{2}, 1] = 1 + 0 \cdot \sqrt{2} + (-1) = 1 - 1 = 0$.

$[1, 0, -1] \cdot [1, -\sqrt{2}, 1] = 0$ & $[1, \sqrt{2}, 1] \cdot [1, -\sqrt{2}, 1] = 0$

(2) In $\mathbb{R}^n$, the set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an orthogonal set; where $\vec{e}_i = [0, \dots \overset{i\text{th}}{1}, 0, \dots, 0]$.

Definition 2) <u>Orthonormal vectors.</u>

A set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ of k-distinct vectors of $\mathbb{R}^n$ is said to be an orthonormal set of vector if and only if it is an orthogonal set and all its vectors are unit vectors, i.e, $\|v_i\| = 1 \ \forall i$.

Example :-(1) In above example, $\{\vec{v}_1 = [1, 0, -1]$, $\vec{v}_2 = [1, \sqrt{2}, 1]$, $\vec{v}_3 = [1, -\sqrt{2}, 1]\}$ is an orthogonal set. The vectors, however, are not orthonormal. Normalizing each vector, we obtain

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right], \quad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \left[\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right]$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \left[\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right].$$

The set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set.

(2) The set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is an orthonormal set in $\mathbb{R}^n$.

Theorem 1 (Orthogonality implies linear independence)
Let $T = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then T is linearly independent.

Proof:- Suppose that

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0} \text{ for some } a_1, \dots a_k \in \mathbb{R}.$$

We need to show that $a_1 = a_2 = a_3 \dots a_k = 0$.

For any index, $1 \le i \le k$, we have.

$$(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k)\cdot\vec{v}_i = \vec{0}.$$

$$\Rightarrow a_1(\vec{v}_1\cdot\vec{v}_i) + a_2(\vec{v}_2\cdot\vec{v}_i) + \dots + a_k(\vec{v}_k\cdot\vec{v}_i) = 0$$

$$\Rightarrow a_1\cdot 0 + a_2\cdot 0 + \dots + a_i\,\vec{v}_i\cdot\vec{v}_i + \dots + a_k\cdot 0 = 0$$

$$\Rightarrow a_i \cdot 1 = 0 \Rightarrow a_i = 0 \quad \left[\because \|\vec{v}_i\| = 1 \Rightarrow \vec{v}_i\cdot\vec{v}_i = 1\right]$$

Orthogonal and Orthonormal basis.

Definition (orthogonal and orthonormal bases)
A basis B for a subspace W of $\mathbb{R}^n$ is said to be an orthogonal basis for W if and only if B is an orthogonal set. Similarly, a basis B for W is an orthonormal basis for W.

if and only if B is an orthonormal set.

<u>Corollary</u> :- If B is an orthogonal set of n non-zero vectors in $\mathbb{R}^n$, then B is an orthogonal basis for $\mathbb{R}^n$. Similarly, if B is an orthonormal set of n vectors in $\mathbb{R}^n$, then B is an orthonormal basis for $\mathbb{R}^n$.

<u>Theorem 2</u> :- Suppose that $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a non-empty ordered orthogonal basis for $\mathbb{R}^n$, and $\vec{v}$ is any vector in $\mathbb{R}^n$. Then.

$$\vec{v} = \frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2} + \frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2} + \cdots + \frac{\vec{v}\cdot\vec{v}_n}{\|\vec{v}_n\|^2}\vec{v}_n.$$

In other words, the coordinatization of $\vec{v}$ with respect to B is

$$[\vec{v}]_B = \begin{bmatrix} \frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2} \\ \frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2} \\ \vdots \\ \frac{\vec{v}\cdot\vec{v}_n}{\|\vec{v}_n\|^2} \end{bmatrix}$$

proof :- Since $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an ordered orthogonal basis for $\mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$, there exist unique real numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that.

$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$.

Now,
$$\vec{v}\cdot\vec{v_i} = (a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n})\vec{v_i}$$
$$= a_1(\vec{v_1}\cdot\vec{v_i}) + a_2(\vec{v_2}\cdot\vec{v_i}) + \cdots + a_i(\vec{v_i}\cdot\vec{v_i}) + \cdots + a_n(\vec{v_n}\cdot\vec{v_i})$$
$$= a_1(0) + a_2(0) + \cdots + a_i\,\vec{v_i}\cdot\vec{v_i} + \cdots + a_n\cdot 0$$
$$= a_i\,\|\vec{v_i}\|^2 \qquad \left[\because v_i\cdot v_j = 0 \text{ if } i\neq j,\ = 1 \text{ if } i=j\right]$$
$$\Rightarrow a_i = \frac{\vec{v}\cdot\vec{v_i}}{\|v_i\|^2}.$$

This completes the proof.

<u>Corollary</u>:- Suppose that $B = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is an ordered orthonormal basis for $\mathbb{R}^n$, and $\vec{v}$ is any vector in $\mathbb{R}^n$. Then
$$\vec{v} = (\vec{v}\cdot\vec{v_1})\vec{v_1} + (\vec{v}\cdot\vec{v_2})\vec{v_2} + \cdots + (\vec{v}\cdot\vec{v_n})\vec{v_n}$$
In other words, the coordinatization of $\vec{v}$ with respect to B is $[v]_B = \begin{bmatrix} v\cdot\vec{v_1} \\ \vec{v}\cdot\vec{v_2} \\ \vdots \\ \vec{v}\cdot\vec{v_n} \end{bmatrix}$

Example :- Verify that the set $B = \{\vec{v_1} = [\frac{1}{\sqrt2}, 0, -\frac{1}{\sqrt2}], \vec{v_2} = [-\frac{1}{3\sqrt2}, \frac{4}{3\sqrt2}, -\frac{1}{3\sqrt2}], \vec{v_3} = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]\}$ is an orthonormal basis for $\mathbb{R}^3$. Then find $[\vec{v}]_B$ for $\vec{v} = [-1, 5, 3]$.

→ The set B is an orthonormal basis for $\mathbb{R}^3$, because it has 3 elements and

$$\vec{v}_1 \cdot \vec{v}_2 = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right] \cdot \left[-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right]$$

$$= -\frac{1}{3} + 0 \cdot \frac{4}{3\sqrt{2}} + \frac{1}{3} = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]\left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right] = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \left[-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right]\left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right] = 0$$

$$\|\vec{v}_1\| = \left\|\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]\right\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0 + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$\|\vec{v}_2\| = 1$ & $\|\vec{v}_3\| = 1$.

Now, $\vec{v} \cdot \vec{v}_1 = [-1, 5, 3]\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]$

$$= (-1)\left(\frac{1}{\sqrt{2}}\right) + 5 \cdot (0) + 3\left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} = \frac{-4}{\sqrt{2}} = \frac{-4\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$= -2\sqrt{2}$$

$$\vec{v} \cdot \vec{v}_2 = [-1, 5, 3]\left[-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right]$$

$$= 3\sqrt{2}.$$

$$\vec{v} \cdot \vec{v}_3 = [-1, 5, 3]\left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right] = 3$$

Thus, by previous corollary, the coordinatization of $\vec{v}$ with respect to B is

$$[\vec{v}]_B = \begin{bmatrix} \vec{v}\cdot\vec{v}_1 \\ \vec{v}\cdot\vec{v}_2 \\ \vec{v}\cdot\vec{v}_3 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ 3\sqrt{2} \\ 3 \end{bmatrix}$$

These coordinates can be verified by checking that

$$\vec{v} = (\vec{v}\cdot\vec{v}_1)\vec{v}_1 + (\vec{v}\cdot\vec{v}_2)\vec{v}_2 + (\vec{v}\cdot\vec{v}_3)\vec{v}_3.$$

The Gram-Schmidt orthogonalization process.

Gram-Schmidt process:

Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. The Gram-Schmidt process constructs a new set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors such that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ becomes an orthogonal basis for span $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$. Following are the steps in Gram-Schmidt process:

Step 1. Let $\vec{v}_1 = \vec{w}_1$.

Step 2. Let $\vec{v}_2 = \vec{w}_2 - \text{proj}_{\vec{v}_1}\vec{w}_2$

$$= \vec{w}_2 - \frac{\vec{w}_2\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1.$$

Step 3. Let $\vec{v}_3 = \vec{w}_3 - \text{proj}_{\vec{v}_1}\vec{w}_3 - \text{proj}_{\vec{v}_2}\vec{w}_3$

$$= w_3 - \frac{(\vec{w}_3 \cdot \vec{v}_1)}{\|\vec{v}_1\|^2} \cdot \vec{v}_1 - \frac{(\vec{w}_3 \cdot \vec{v}_2)}{\|\vec{v}_2\|^2} \cdot \vec{v}_2 \qquad (92)$$

Continue this process upto $\vec{v}_k$. Then the resulting set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ forms an orthogonal basis for the subspace spanned by the set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$.

Theorem :- Let $B = \{\vec{w}_1, \vec{w}_2 \ldots \vec{w}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ obtained by applying the Gram - Schmidt process to B is an orthogonal basis for W.

Example :- Consider the subset $B = \{[2,1,0,-1], [1,0,2,-1], [0,-2,1,0]\}$ of $\mathbb{R}^4$. Using the independence Test Method, we have to show that B is L.I.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$[A|0] = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$\xrightarrow[R_4 \to R_4 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$ (93)

$\xrightarrow{R_3 \to -\frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow[R_2 \to R_2 - 4R_3]{R_1 \to R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

$\therefore$ B is L.I. Let $W = \text{Span}(B)$.

Now, B is not an orthogonal basis for W, because $[2,1,0,-1]\,[1,0,2,1] = 3 \neq 0$.

We will apply the Gram-Schmidt process to find an orthogonal basis for W. Let

$\vec{w}_1 = [2,1,0,-1]$, $\vec{w}_2 = [1,0,2,-1]$ and

$\vec{w}_3 = [0,-2,1,0]$.

Following the Gram-Schmidt process,

we set $\vec{v}_1 = \vec{w}_1 = [2,1,0,-1]$.

Next, we have $\vec{v}_2 = \vec{w}_2 - \left(\frac{\vec{w}_2 \cdot \vec{v}_1}{\|w_1\|^2}\right)\vec{v}_1$.

$$= [1,0,2,-1] - \frac{[1,0,2,-1]\cdot[2,1,0,-1]}{\|[2,1,0,-1]\|^2}[2,1,0,-1]$$

$$= [1,0,2,-1] - \left(\frac{3}{6}\right)[2,1,0,-1]$$

$$= [1,0,2,-1] - \left[1,\frac{1}{2},0,-\frac{1}{2}\right]$$

$$= \left[0,-\frac{1}{2},2,-\frac{1}{2}\right]$$

Multiplying this vector by 2 to avoid fractions, we get $\vec{v}_2 = [0,-1,4,-1]$. Notice that $\vec{v}_2$ is orthogonal to $\vec{v}_1$.

Finally,

$$\vec{v}_3 = \vec{w}_3 - \left(\frac{\vec{w}_3\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{w}_3\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2$$

$$= [0,-2,1,0] - \frac{[0,-2,1,0]\cdot[2,1,0,-1]}{\|[2,1,0,-1]\|^2}$$

$$[2,1,0,-1] - \frac{[0,-2,1,0][0,-1,4,-1]}{\|[0,-1,4,-1]\|^2}[0,-1,4,-1]$$

$$= [0,-2,1,0] - \left(\frac{-2}{6}\right)[2,1,0,-1] - \left(\frac{6}{18}\right)[0,-1,4,-1]$$

$$= [0,-2,1,0] - \left[\frac{2}{3},-\frac{1}{3},0,\frac{1}{3}\right] - \left[0,-\frac{1}{3},\frac{4}{3},-\frac{1}{3}\right]$$

$$= \left[\frac{2}{3},-\frac{4}{3},-\frac{1}{3},0\right]$$

To avoid functions, we multiply this vector by 3 to get $\vec{v}_3 = [2,-4,-1,0]$. Notice that $\vec{v}_3$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$.

Orthogonal Complements.

Definition (Orthogonal Complement).

Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W, denoted by $W^\perp$, is the set of all vectors of $\mathbb{R}^n$ that are orthogonal to every vector in W. That is,

$$W^\perp = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}$$

Theorem :- If W is a subspace of $\mathbb{R}^n$, then $\vec{v} \in W^\perp$ if and only if $\vec{v}$ is orthogonal to every vector in a spanning set for W.

proof:- Let $S = \{ \vec{w}_1, \vec{w}_2, \dots, \vec{w}_k \}$ be a spanning set for W i.e. $W = \text{Span}(S)$. First, we assume that $v \in W^\perp$. We must show that $\vec{v} \cdot \vec{w}_i = 0$ for $1 \le i \le k$. Because $\vec{v} \in W^\perp$, we have $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$.

In particular, $\vec{v} \cdot \vec{w}_i = 0$ for $1 \le i \le k$,

since $S \subseteq \text{span}(S) = W$.

Conversely, Suppose that $\vec{v} \cdot \vec{w}_i = 0$ for $1 \le i \le k$.

Let $\vec{w} \in W = \text{Span}(S)$. Then there exist real numbers $a_1, a_2, \ldots, a_k$ such that

$$\vec{w} = a_1 \vec{w_1} + a_2 \vec{w_2} + \cdots + a_k \vec{w_k}$$

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot (a_1 \vec{w_1} + a_2 \vec{w_2} + \cdots + a_k \vec{w_k})$$

$$= a_1 (\vec{v} \cdot \vec{w_1}) + a_2 (\vec{v} \cdot \vec{w_2}) + \cdots + a_k (\vec{v} \cdot \vec{w_k})$$

$$= 0$$

$$\Rightarrow \vec{v} \cdot \vec{w} = 0.$$

$$\Rightarrow \vec{v} \in W^{\perp}.$$

Example :- (1) Consider $V = \mathbb{R}^n$ & $S = \{\vec{e_1}, \ldots \vec{e_n}\}$.

Then $\mathbb{R}^n = \text{Span}(S)$.

$$(\text{Span}(S))^{\perp} = (\mathbb{R}^n)^{\perp} = \{\vec{0}\}$$

$$\therefore (\mathbb{R}^n)^{\perp} = \{\vec{0}\}.$$

(2) Consider $W = \{[a, 0, 0] \mid a \in \mathbb{R}\}$ be a subspace of $\mathbb{R}^3$.

$$W = \text{Span}\{[1, 0, 0]\}.$$

By previous theorem, $[x, y, z] \in W^{\perp} \Leftrightarrow$

$$[x, y, z] \cdot [1, 0, 0] = 0.$$

$\Rightarrow$ $[x, y, z] \in W^{\perp} \Leftrightarrow$ ~~[illegible]~~ $x = 0$

$W^{\perp} = \{ [0, y, z] \mid x, y \in \mathbb{R} \}$

$= \{ y[0,1,0] + z[0,0,1] \mid x, y \in \mathbb{R} \}$

$= \text{span} \{ [0,1,0], [0,0,1] \}$.

Clearly, $\dim(W) = 1$ and $\dim(W^{\perp}) = 2$.

$\dim(W) + \dim(W^{\perp}) = 1 + 2 = 3 = \dim(\mathbb{R}^3)$

$\therefore \underline{\dim(W) + \dim(W^{\perp}) = \dim(\mathbb{R}^3)}$.

(3) Consider $V = \mathbb{R}^3$ & $W = \{ a[-1, 3, 2] \mid a \in \mathbb{R} \}$

$[x, y, z] \in W^{\perp} \Leftrightarrow [x, y, z] \cdot [-1, 3, 2] = 0$

$\Leftrightarrow -x + 3y + 2z = 0$.

$\therefore W^{\perp} = \{ [x, y, z] \mid -x + 3y + 2z = 0 \}$.

Choose ~~putting~~ $y = 1$ and $z = 0$, we get

$x = 3$.

Choose $y = 0$ and $z = 1$, we get

$x = 2$.

$W^{\perp} = \text{span} \{ [3, 1, 0], [2, 0, 1] \}$.

Clearly, $\dim W = 1$ & $\dim W^{\perp} = 2$.

we have

$$\dim(W) + \dim(W^{\perp}) = 1 + 2 = 3 = \dim(\mathbb{R}^3)$$

(8) prove that the set $\{\vec{v}_1 = [1, 0, -1], \vec{v}_2 = [-1, 4, -1], \vec{v}_3 = [2, 1, 2]\}$ is an orthogonal set but not orthonormal set.

→

$$\vec{v}_1 \cdot \vec{v}_2 = [1, 0, -1][-1, 4, -1]$$

$$= -1 + 0 + 1 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = [1, 0, -1][2, 1, 2] = 2 + 0 - 2 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = [-1, 4, -1][2, 1, 2] = -2 + 4 - 2 = 0.$$

Therefore, the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.

$$\|\vec{v}_1\| = \|[1, 0, -1]\| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \neq 1$$

Therefore, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not orthonormal.

(9) Verify that the set

$B = \{\vec{v}_1 = [\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}], \vec{v}_2 = [\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}], \vec{v}_3 = [\frac{6}{7}, \frac{2}{7}, \frac{3}{7}]\}$. Then find $[v]_B$ for the vector $\vec{v} = [5, -1, 2]$.

(99)

→ The set B is an orthonormal basis for $\mathbb{R}^3$, because it has 3 elements and

$$\vec{v_1}\cdot\vec{v_2} = \left[\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}\right]\left[\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right]$$

$$= \frac{6}{7} - \frac{18}{7} + \frac{12}{7} = \frac{18}{7} - \frac{18}{7} = 0$$

$$\vec{v_2}\cdot\vec{v_3} = \left[\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right]\left[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right]$$

$$= \frac{12}{7} + \frac{6}{7} - \frac{18}{7} = \frac{18}{7} - \frac{18}{7} = 0$$

$$\vec{v_1}\cdot\vec{v_3} = \left[\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}\right]\left[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right]$$

$$= \frac{18}{7} - \frac{12}{7} - \frac{6}{7} = \frac{18}{7} - \frac{18}{7} = 0$$

$$\|\vec{v_1}\| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9+36+4}{49}} = \sqrt{\frac{49}{49}} = 1$$

Silly, $\|\vec{v_2}\| = 1 = \|\vec{v_3}\|$

Now $\vec{v}\cdot\vec{v_1} = [4, -1, 2]\left[\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}\right]$

$$= \frac{12}{7} + \frac{6}{7} - \frac{4}{7}$$

$$= \frac{18-4}{7} = \frac{14}{7} = 2$$

$W = \operatorname{span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$

$\dim W + \dim(W^\perp) = \dim(\mathbb{R}^3)$

$\Rightarrow 3 + 0 = 3$.

$\therefore \dim(W^\perp) = 0$

$\therefore \dim W = \dim \mathbb{R}^3$

$\therefore W$ is a basis for $\mathbb{R}^3$.

$\vec{v}\cdot\vec{v_2} = [4, -1, 2]\left[\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right]$

$$= \frac{8}{7} - \frac{3}{7} - \frac{12}{7} = \frac{8-15}{7} = -\frac{7}{7} = -1$$

$\vec{v}\cdot\vec{v_3} = [4, -1, 2]\left[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right]$

$$= \frac{24}{7} - \frac{2}{7} + \frac{6}{7} = \frac{30-2}{7} = \frac{28}{7} = 4$$

Scanned with CamScanner

The co-ordinatization of $\vec{v}$ with respect to B is

given by

$$[\vec{v}]_B = \begin{bmatrix} \vec{v}\cdot\vec{v}_1 \\ \vec{v}\cdot\vec{v}_2 \\ \vec{v}\cdot\vec{v}_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

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