Weekly e-material (23/12/2020 to 27/12/2020) (P.1)

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The matrix of a linear transformation

Theorem 1 (A linear transformation is determined by its action on a basis)

Let $B = \{v_1, \ldots, v_n\}$ be an ordered basis for a vector space V. Let W be a vector space, and let $\{w_1, w_2, \ldots, w_n\}$ be any n (not necessarily distinct) vectors in W. Then there is one and only one linear transformation $T : V \rightarrow W$ satisfying $T(v_1) = w_1$, $T(v_2) = w_2, \ldots, T(v_n) = w_n$.

Proof:- Let v be any vector in V. Since $B = \{v_1, \ldots, v_n\}$ is an ordered basis for V, there exist unique scalars $a_1, a_2, \ldots, a_n$ in $\mathbb{R}$ such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Define a function $T : V \longrightarrow W$ by

$$T(v) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$$

Let $v = \sum_{i=1}^{n} a_i v_i$ & $v' = \sum_{i=1}^{n} a_i' v_i \in V$ such that

$$v = v' \Rightarrow$$

Then $a_i = a_i'$ for all i ($\because \{v_1, \ldots, v_n\}$ is L.I.)

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we have

$$\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} a_i w_i$$

$$\Rightarrow T(v) = T(v').$$

$\therefore$ T is well-defined.

We will show that T is a linear transformation.

Let v and v' be two vectors in V. Then

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$

$$\& \quad v' = b_1' v_1 + b_2' v_2 + \cdots + b_n' v_n,$$

where b_i & $b_i' \in \mathbb{R}$ for all i.

By definition of T, we have.

$$T(v) = b_1 w_1 + b_2 w_2 + \cdots + b_n w_n.$$

$$\& \quad T(v') = b_1' w_1 + b_2' w_2 + \cdots + b_n' w_n.$$

$$T(v + v') = T\big((b_1 v_1 + b_2 v_2 + \cdots + b_n v_n) + (b_1' v_1 + b_2' v_2 + \cdots + b_n' v_n)\big)$$

$$= T\big((b_1 + b_1')v_1 + (b_2 + b_2')v_2 + \cdots + (b_n + b_n')v_n\big)$$

$$= (b_1 + b_1')w_1 + (b_2 + b_2')w_2 + \cdots + (b_n + b_n')w_n$$

$$= (b_1 w_1 + b_2 w_2 + \cdots + b_n w_n) + (b_1' w_1 + b_2' w_2 + \cdots + b_n' w_n)$$

$$= T(v) + T(v').$$

Let $c \in \mathbb{R}$. Then

$$\& \quad T(cv) = T\big(c(b_1 v_1 + b_2 v_2 + \cdots + b_n v_n)\big)$$

$$= T\big((cb_1)v_1 + (cb_2)v_2 + \cdots + (cb_n)v_n\big)$$

$$= (cb_1)w_1 + (cb_2)w_2 + \cdots + (c.b_n)w_n$$

$$= c(b_1 w_1 + b_2 w_2 + \cdots + b_n w_n) = cT(v).$$

Hence T is a linear Transformation.

Now, to prove the uniqueness, let $T' : V \to W$ be another linear transformation satisfying

$$T'(v_1) = w_1,\ T'(v_2) = w_2, \dots, T'(v_n) = w_n.$$

If $v \in V$, then $v = a_1 v_1 + \dots + a_n v_n$, $a_i \in \mathbb{R}$. Then.

$$T'(v) = T'(a_1 v_1 + \dots + a_n v_n)$$
$$= a_1 T'(v_1) + \dots + a_n T'(v_n).$$
$$= a_1 w_1 + \dots + a_n w_n.$$
$$= T(v) \text{ for all } v \in V$$

$\Rightarrow T' = T$ and hence T is uniquely determined

Example 1) Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with $L([1,0,0]) = [-3,2,4]$, $L([0,1,0]) = [5,-1,3]$ and $L([0,0,1]) = [-4,0,-2]$. Find $L([x,y,z])$, for any $[x,y,z] \in \mathbb{R}^3$ and also find $L([6,2,-7])$.

→ We have to find

$$L([x,y,z]) \text{ for any } [x,y,z] \in \mathbb{R}^3.$$

Since $\{\vec{e}_1 = [1,0,0], \vec{e}_2 = [0,1,0], \vec{e}_3 = [0,0,1]\}$ are basis for $\mathbb{R}^3$. Then

$$[x,y,z] = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$$

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It is given that
$L([1,0,0]) = [-3,2,4]$, $L([0,1,0]) = [5,-1,3]$ &
$L([0,0,1]) = [-4,0,-2]$.

Then by Theorem 1, we have

$$L([x,y,z]) = L(x\vec{e_1} + y\vec{e_2} + z\vec{e_3})$$
$$= x[-3,2,4] + y[5,-1,3] + z[-4,0,-2]$$
$$= [-3x+5y-4z, 2x-y, 4x+3y-2z]$$

Now, we have to find $L([6,2,-7])$.

$$L([6,2,-7]) = [-3.6+5.2-4(-7), 2.6-2.2, 4.6+3.2-2(-7)]$$
$$= [20, 10, 44]$$

(2) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L([1,1]) = [1,3]$ and $L([-2,3]) = [-4,3]$. Express $L([1,0])$ and $L([0,1])$ as linear combinations of the vectors $[1,0]$ and $[0,1]$.

$\rightarrow$ To find $L([0,1])$ and $L([1,0])$, we first express the vectors $[1,0]$ and $[0,1]$ as linear combinations of vectors $[1,1]$ and $[-2,3]$.

Let $[1,0] = a_1[1,1] + b_1[-2,3]$ &
$[0,1] = a_2[1,1] + b_2[-2,3]$

$\Rightarrow$ ~~$a_1+2b_1=1$~~

$\Rightarrow$ $a_1 - 2b_1 = 1$
$a_1 + 3b_1 = 0 \rightarrow (1)$

& $a_1 - 2b_1 = 0 \rightarrow (2)$
$a_1 + 3b_1 = 1$

To solve eqns. (1) & (2), we row reduce the augmented matrix.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \end{array}\right]$$

Therefore $[1, 0] = \frac{3}{5}[1,1] - \frac{1}{5}[-2, 3]$

& $[0, 1] = \frac{2}{5}[1,1] + \frac{1}{5}[-2, 3]$.

This gives,
$$L([1,0]) = L\left(\frac{3}{5}[1,1] - \frac{1}{5}[-2,3]\right)$$
$$= \frac{3}{5}L([1,1]) - \frac{1}{5}L([-2,3])$$
$$= \frac{3}{5}[1,-3] - \frac{1}{5}[-4,2]$$

$$= \left[\frac{7}{5}, \frac{-11}{5}\right] = \frac{1}{5}[7, -11]$$

$$= \frac{7}{5}[1,0] + \left(-\frac{11}{5}\right)[0,1]$$

$$= \frac{7}{5}[1,0] - \frac{11}{5}[0,1]$$

& $$L([0,1]) = L\left(\frac{2}{5}[1,1] + \frac{1}{5}[-2,3]\right)$$

$$= \frac{2}{5}L([1,1]) + \frac{1}{5}L([-2,3])$$

$$= \frac{2}{5}([1,-3]) + \frac{1}{5}([-4,3])$$

$$= \left[-\frac{2}{5}, \frac{-4}{5}\right]$$

$$= -\frac{2}{5}[1,0] - \frac{4}{5}[0,1]$$

(#) The matrix of a linear Transformation.

Let V and W be non-trivial vector spaces, with $\dim V = n$ and $\dim W = m$. Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$ be ordered bases for V and W, respectively. Let $T : V \to W$ be a linear transformation. For each $v \in V$, the coordinate vectors for v and $T(v)$ with respect to ordered bases B and C are $[v]_B$ and $[T(v)]_C$, respectively. Our aim is to find an $m \times n$ matrix $A = [a_{ij}]$ $\left(1 \le i \le m;\ 1 \le j \le n\right)$

Such that

$$A[v]_B = [T(v)]_C \quad \text{--- (1)}$$

holds for all $v \in V$. Since eq^n. (1) must hold for all vectors in V, it must hold, in particular, for the basis vectors in B, that is,

$$A[v_i]_B = [T(v_i)]_C \text{ for all } 1 \le i \le n. \text{ --- (2)}$$

But, $v_1 = 1.v_1 + 0.v_2 + \cdots + 0.v_n$

$\vdots$

$v_n = 0.v_1 + 0.v_2 + \cdots + 1.v_n$

Therefore, $[v_1]_B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $[v_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ - - -

$[v_n]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

$$A[v_1]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Silly, $A[v_2]_B = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, A[v_n]_B = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

Substituting these results into eq^n. (2), we obtain

$$[T(v_1)]_C = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [T(v_2)]_C = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots$$

$$[T(v_n)]_C = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This shows that the successive columns of A are the coordinate vectors of $T(v_1), T(v_2), \ldots, T(v_n)$ with respect to the ordered basis C. Thus, the matrix A is given by

$$A = \Big[[T(v_1)]_C \; [T(v_2)]_C \cdots [T(v_n)]_C \Big]$$

We will call this matrix as the matrix of T relative to the bases B and C and will denote it by the symbol A_{BC} ~~or~~ $[T]_{BC}$.

Thus,

$$A_{BC} = \Big[[T(v_1)]_C \; [T(v_2)]_C \cdots [T(v_n)]_C \Big]$$

From eqn. (1), the matrix A_{BC} satisfies the property

$$A_{BC}[v]_B = [T(v)]_C \quad \text{for all } v \in V$$

We have thus proved the following theorem:

<u>Theorem 2</u> :- Let V and W be non-trivial vector spaces, with $\dim(W) = m$. Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$ be ordered bases for V and W, respectively. Let $T : V \to W$ be a linear transformation. Then there is a unique $m \times n$ matrix A_{BC} such that $A_{BC}[v]_B = [T(v)]_C$ for all $v \in V$.

Furthermore, for $1 \leq i \leq n$, the ith column of

$$A_{BC} = [T(v_i)]_C .$$

Example ①:- Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T([x, y, z]) = [-2x_1 + 3x_3, x_1 + 2x_2 - x_3].$$

Find the matrix for T with respect to the ordered bases $B = \{[1,-3,2], [-4,3,-3], [2,-3,2]\}$ for $\mathbb{R}^3$ and $C = \{[-2,-1], [5,3]\}$ for $\mathbb{R}^2$.

→ We have

$$T([1,-3,2]) = [-2.1 + 3.2, 1 + 2.(-3) - 2]$$
$$= [4, -7]$$

$$T([-4,3,-3]) = [-2(-4) + 3.(-3), -4 + 2.3 - (-3)]$$
$$= [-1, 5]$$

& $T([2,3,2]) = [2,-6,-6]$

The coordinate vectors of these vectors with respect to the ordered basis $C = \{[-2,-1], [5,3]\}$ for $\mathbb{R}^3$ are given by

$$T([1,-3,2]) = a_1[-2,-1] + b_1[5,3]$$

$$\Rightarrow [4,-7] = [-2a_1 + 5b_1, -a_1 + 3b_1]$$

$$\Rightarrow \left.\begin{array}{l} -2a_1 + 5b_1 = 4 \\ -a_1 + 3b_1 = -7. \end{array}\right\} \rightarrow (1)$$

To find the coordinates of $T[1,-3,2]$ with respect to the ordered basis $C = \{[-2,-1], [5,3]\}$, we used the Coordinatization Method on these vectors, i.e, we row reduce the augmented matrix.

$$\left[\begin{array}{cc|c} -2 & 5 & 4 \\ -1 & 3 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & 3 & -7 \\ -2 & 5 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|c} 1 & -3 & 7 \\ -2 & 5 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & -1 & 18 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} 1 & -3 & 7 \\ 0 & 1 & -18 \end{array}\right] \xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cc|c} 1 & 0 & -47 \\ 0 & 1 & -18 \end{array}\right]$$

$$\therefore \left[T([1,-3,2])\right]_C = \begin{bmatrix} -47 \\ -18 \end{bmatrix}$$

Sily, we determine the coordinates of $T([-4,3,-3])$ and $T([2,3,2])$ with respect to the ordered basis B, we now reduce the augmented matrixes

$$\left[\begin{array}{cc|c} -2 & 5 & -1 \\ -1 & 3 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & 3 & 5 \\ -2 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ -2 & 5 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & -1 & -11 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 1 & +1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 28 \\ 0 & 1 & 11 \end{array}\right]$$

$$\therefore \left[T([-4,3,-3])\right]_C = \begin{bmatrix} 28 \\ 11 \end{bmatrix}$$

& sily, $$\left[T([2,3,2])\right]_C = \begin{bmatrix} -36 \\ -14 \end{bmatrix}$$

Thus the matrix A_{BC} to relative to the bases B and C is given by

$$A_{BC} = \Big[[T([1,-3,2])]_C \; [T([-4,3,-3])]_C \; [T([2,3,2])]_C \Big]$$

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$$A_{BC} = \begin{bmatrix} -47 & 28 & -36 \\ -18 & 11 & -14 \end{bmatrix}$$

(2). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T([x, y, z]) = [-6x+4y-z, -2x+3y-5z, 3x-y+7z]$.

Find the matrix for T with respect to the standard basis for $\mathbb{R}^3$.

$\to$

$$T(\vec{e_1}) = T([1,0,0]) = [-6, -2, 3]$$

$$T(\vec{e_2}) = T([0,1,0]) = [4, 3, -1]$$

$$T(\vec{e_3}) = T([0,0,1]) = [-1, -5, 7]$$

We have.

$$T(\vec{e_1}) = (-6)\vec{e_1} + (-2)\vec{e_2} + 3\vec{e_3}$$

$$T(\vec{e_2}) = 4\vec{e_1} + 3\vec{e_2} + (-1)\vec{e_3}$$

$$\& \; T(\vec{e_3}) = (-1)\vec{e_1} + (-5)\vec{e_2} + 7\vec{e_3}.$$

We get

$$[T(\vec{e_1})]_B = \begin{bmatrix} -6 \\ -2 \\ 3 \end{bmatrix}, [T(\vec{e_2})]_B = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

$$[T(\vec{e_3})]_B = \begin{bmatrix} -1 \\ -5 \\ 7 \end{bmatrix}, \text{ where } B = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$$

Therefore,

$$A_{BB} = \begin{bmatrix} -6 & 4 & -1 \\ -2 & 3 & -5 \\ 3 & -1 & 7 \end{bmatrix}$$

③ Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by $T([x_1, x_2, x_3]) = [-2x_1 + 3x_3, x_1 + 2x_2 - x_3]$. Find the matrix for T with respect to the ordered bases $B = \{[1, -3, 2], [-4, 13, -3], [2, -3, 20]\}$ for $\mathbb{R}^3$ and $C = \{[-2, -1], [5, 3]\}$ for $\mathbb{R}^2$.

→ We have to compute the matrix A_{BC} of T with respect to the ordered bases B and C is given.

$$A_{BC} = \left[[T(\vec{v}_1)]_C, [T(\vec{v}_2)]_C, [T(\vec{v}_3)]_C \right],$$

where $\vec{v}_1 = [1, -3, 2]$, $\vec{v}_2 = [-4, 13, -3]$ &

$\vec{v}_3 = [2, -3, 20]$.

To determine the coordinates of $[T(\vec{v}_1)]_C$, $[T(\vec{v}_2)]_C$ and $[T(\vec{v}_3)]_C$ with respect to the ordered basis B, we now reduce the augmented matrices. (By Coordinatitation Method)

$$[\vec{w}_1 \ \vec{w}_2 \mid T(\vec{v}_1)]$$

$$[\vec{w}_1 \ \vec{w}_2 \mid T(\vec{v}_2)]$$

& $[\vec{w}_1 \ \vec{w}_2 \mid T(\vec{v}_3)]$, where $\vec{w}_1 = [-2, -1]$ & $\vec{w}_2 = [5, 3]$

(43)

- It is similar to row reduce the augmented matrix

$$\left[\begin{array}{cc|ccc} \vec{w}_1 & \vec{w}_2 & T(v_1) & T(v_2) & T(v_3) \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ -1 & 3 & -7 & 25 & -24 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|ccc} -1 & 3 & -7 & 25 & -24 \\ -2 & 5 & 4 & -1 & 56 \end{array}\right]$$

$$\xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|ccc} 1 & -3 & 7 & -25 & 24 \\ -2 & 5 & 4 & -1 & 56 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|ccc} 1 & -3 & 7 & -25 & 24 \\ 0 & -1 & 18 & -51 & 104 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|ccc} 1 & -3 & 7 & -25 & 24 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cc|ccc} 1 & 0 & -47 & 128 & -288 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right]$$

Therefore

$$\left[T(\vec{v}_1)\right]_C = \begin{bmatrix} -47 \\ -18 \end{bmatrix}$$

$$[T(\vec{v}_2)]_C = \begin{bmatrix} 128 \\ 51 \end{bmatrix} \;\&\; [T(\vec{v}_3)]_C = \begin{bmatrix} -288 \\ -104 \end{bmatrix}.$$

we get,

$$A_{BC} = \begin{bmatrix} -47 & 128 & -288 \\ -18 & 51 & -104 \end{bmatrix}$$

Finding the New Matrix for a Linear Transformation After a Change of Bases

Theorem 3 :- Let V and W be two non-trivial finite-dimensional vector spaces with ordered bases B and C. Suppose that D and E are other ordered bases for V and W, respectively. Let P be the transition matrix from B to D, and let Q be the transition matrix from C to E. Then the matrix A_{DE} for T with respect to bases D and E is given by

$$A_{DE} = Q A_{BC} P^{-1}.$$

Example (1) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T([a,b,c]) = [-2a+b, -b-c, a+3c]$

(a) Find the matrix A_{BB} for T with respect to the standard basis $B = \{\vec{e}_1 = [1,0,0], \vec{e}_2 = [0,1,0], \vec{e}_3 = [0,0,1]\}$ for $\mathbb{R}^3$.

(b) Use part (a) to find the matrix A_{DE} (45)

with respect to the standard bases

$D = \{[15,-6,4], [2,0,1], [3,-1,1]\}$

and $E = \{[1,-3,1], [0,3,-1], [2,-2,1]\}$.

→ (a) We have

$T(\vec{e_1}) = [-2,0,1]$, $T(\vec{e_2}) = [1,-1,0]$.

& $T(\vec{e_3}) = [0,-1,3]$.

We get

$$A_{BB} = \left[[T(\vec{e_1})]_B \;\; [T(\vec{e_2})]_B \;\; [T(\vec{e_3})]_B \right]$$

$$= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad B = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$$

(b) We find

$$A_{DE} = Q A_{BB} \cdot P^{-1};$$

Where Q is the transition matrix from B to E. (i.e. $Q_{E \leftarrow B}$) and P is the transition matrix from B to D (i.e. $P_{D \leftarrow B}$).

Since P^{-1} is the transition matrix from D to B.

$$P^{-1} = \left[[15,-6,4]_D \cdot [2,0,1]_D \; [3,-1,1]_D \right]$$

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$$= \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

Hence using eqⁿ(1), we obtain.

$$A_{DE} = Q A_{BB} P^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -202 & -32 & -43 \\ -146 & -23 & -31 \\ 83 & 14 & 18 \end{bmatrix}$$

Linear operators and Similarity.

Theorem : Let V be a finite-dimensional vector space with ordered bases B and C. Let T be a linear operator on V. Then the matrix A_{BB} for T with respect to the basis B is similar to the matrix A_{CC} for T with respect to the basis C. More specifically, if P is the transition matrix from B to C, then

$$A_{BB} = P^{-1} A_{CC} P.$$

Example :- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $T([x, y]) = [2x - y, x - 3y]$. Find the matrix for T with respect to the basis $\{[4,-1], [-7,2]\}$ using the method of similarity.

→ We have

$T(\vec{e_1}) = T([1,0]) = [2,1]$.

$T(\vec{e_2}) = T([0,1]) = [-1,-3]$

Thus, the matrix for T with respect to the $B = \{\vec{e_1}, \vec{e_2}\}$ is $A_{BB} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$.

Next, we find the transition matrix P from the standard basis B to the basis $C = \{[4,-1], [-7,2]\}$.

To find P, we first find P^{-1}, the transition matrix from C to B, which is given by

$$P^{-1} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$$

Hence the matrix A_{CC} for T with respect to the basis C is given by

$$A_{CC} = P A_{BB} P^{-1} = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & -123 \\ 37 & -68 \end{bmatrix}$$

The Dimension Theorem

Definition (Kernel and Range of a linear Transformation)

Let V and W be two vector spaces, and let $T: V \to W$ be a linear transformation. Then the kernel (also called the null space) of T, denoted by $ker(T)$, is the set of all $v \in V$ such that $T(v) = O_W$. That is,

$$ker(T) = \{ v \in V \mid T(v) = O_W \}.$$

The range of T, denoted by range (T), is the set of all vectors in W that are the image of some vector in V. ~~that~~ That is,

$$range(T) = \{ w \in W \mid w = T(v) \text{ for some } v \in V \}$$

Theorem :- Let $T: V \to W$ be ~~a~~ linear transformation. Then $ker(T)$ is a subspace of V and range(T) is a subspace of W.

proof :- By definition,

$$ker(T) = \{ v \in V \mid T(v) = O_W \}$$

$$= T^{-1}(\{O_W\})$$

$$\& \ range(T) = T(V)$$

We already shown that $ker(T)$ and range(T) are subspaces.

of V and W, respectively.

Theorem :- Suppose that $T: V \to W$ is a linear transformation. Show that if $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is a linearly independent set of k distinct vectors in W, for some vectors $v_1, \ldots, v_k \in V$, then $\{v_1, v_2, \ldots, v_k\}$ is a L.I. set in V.

proof:- Suppose that

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0_v, \quad a_1, a_2, \ldots, a_k \in \mathbb{R}$$

$$\Rightarrow T(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) = T(0_v)$$

$$\Rightarrow a_1 T(v_1) + a_2 T(v_2) + \cdots + a_k T(v_k) = 0_W$$

Since $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ are L.I.

Therefore, $a_1 = a_2 = \cdots = a_k = 0$.

This completes the proof.

Method for finding a basis for the kernel of a linear transformation

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(X) = AX$, where A is a fixed $m \times n$ matrix and $X \in \mathbb{R}^n$. Now, $\ker(T)$ is the subspace consisting of all vectors X in the domain $\mathbb{R}^n$ that are solutions of the homogeneous system $AX = O$. We now outline the procedure to find a basis for $\ker(T)$.

Step 1. Find B, the row reduced echelon form of A

Step 2. Obtain a set $\{v_1, v_2, \ldots, v_n\}$ of a particular solutions of the homogeneous system $BX = O$ by setting each independent variable in turn equal to 1 and all other independent variables equal to 0.

Step 3: The set $\{v_1, \ldots, v_n\}$ is a basis for $\ker(T)$. We can eliminate fractions by replacing any v_i with cv_i, where $c \neq 0$.

Examples (1) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use the Kernel Method to find a basis for $\ker(T)$.

→ We first B, the row reduced echelon form of A, where $A = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 0 & 1 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 4 \\ 0 & 6 & -6 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 6 & -6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 6R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the reduced row echelon form of A. has no non-pivot column.

Hence the homogeneous system $BX = 0$ has unique solution.

Therefore $\ker(T) = \{0\}$.

(2) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 & -3 \\ 2 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use the Kernel Method to find a basis for $\ker(T)$.

$\rightarrow$ We first row reduce $A = \begin{bmatrix} 3 & 1 & -3 \\ 2 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix}$ as follows:

$$\begin{bmatrix} 3 & 1 & -3 \\ 2 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{7}{3} & 7 \end{bmatrix} \xrightarrow{\cancel{R_3 \to R_3 - 7R_2}} \begin{bmatrix} \cancel{0} & \cancel{\frac{1}{3}} & \cancel{-1} \\ 0 & \frac{1}{3} & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

~~The~~ $\xrightarrow{R_2 \to 3R_2} \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 0 & 1 & 3 \\ 0 & \frac{7}{3} & 7 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{7}{3}R_2} \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_1 \to R_1 - \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix B in reduced row echelon form of A has one non-pivot column, namely, Column 3. Hence the homogeneous system

$BX = 0$ has one independent variable x_3 &

and

$$x_1 - 2x_3 = 0$$

$$x_2 + 3x_3 = 0$$

i.e. $x_1 = 2x_3$

$x_2 = -3x_3$.

Setting $x_3 = 1$ gives a particular solution $[2, -3, 1]$. The set $\{[2, -3, 1]\}$ forms a basis for $\ker(T)$.

Method for finding a Basis for the Range of a Linear Transformation.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $T(X) = AX$, where A is a fixed $m \times n$ matrix and $X \in \mathbb{R}^n$. We now outline the procedure for finding a basis for the range of T.

Step 1. Find B, the row reduced echelon form of A.

Step 2. From the set of those columns of A whose corresponding columns in B have non-zero pivots. The set so formed is a basis for range (T).

Example (1) :- Consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(X) = AX$,

$$A = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Use the Range Method to find a basis for range (T). Also find a basis for ker(T), and verify that

$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(\mathbb{R}^3)$$

→ From above example 1, we get the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the reduced row echelon form of A having non-zero pivots in columns 1, 2 and 3.

Hence the corresponding columns of A form a basis for range(T). In other words, the set $\{[5,-3,1], [1,0,1], [-1,1,1]\}$ forms a basis for range(T).

and so $\dim(\text{range}(T)) = 3$.

Also, $\ker(T) = \{0\}$, $\dim(\ker(T)) = 0$.

clearly
$$\dim(\text{range}(T)) + \dim(\ker(T))$$
$$= 3 + 0 = 3 = \dim(\mathbb{R}^3).$$

(2) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x) = AX$, where

$$A = \begin{bmatrix} 3 & 1 & -3 \\ 2 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix}$$

$\Rightarrow$ From above example 2, we get the matrix $B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is the reduced row echelon form of A having non-zero pivots in columns 1 and 2. Hence the corresponding columns of A form a basis for range(T). In other words, the set $\{[3,2,2], [1,1,3]\}$ forms a basis for range(T), and so $\dim(\text{range}(T)) = 2$.

In example 2, we know that

~~ker(T) = { [~~

the set $\{[2,-3,1]\}$ forms a basis for $\ker(T)$. Therefore $\dim(\ker(T)) = 1$.

Furthermore, $\dim(\text{range}(T)) + \dim(\ker(T))$

$$= 2 + 1$$
$$= 3 = \dim(\mathbb{R}^3).$$

The dimension theorem.

<u>Theorem</u> (Dimension Theorem):

If $T: V \to W$ is a linear transformation and V is finite-dimensional, then range (T) is finite-dimensional, and
$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim V.$$

Example 1) Consider the linear transformation ~~T([x~~. $T: \mathbb{R}^4 \to \mathbb{R}^4$ given by
$$T([x_1, x_2, x_3, x_4]) = [x_1, 0, x_3, 0].$$

Find a basis for $\ker(T)$ and a basis for range (T), and verify that

$\dim(\ker(T)) + \dim(\text{range}(T)) =$ (57)

$\dim(\mathbb{R}^4)$

$\rightarrow \ker(T) = \{ [x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid T([x_1, x_2, x_3, x_4]) = [0,0,0,0] \}$

$= \{ [x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid [x_1, 0, x_3, 0] = [0,0,0,0] \}$

$= \{ [x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid x_1 = 0 = x_3 \}$

$= \{ [0, x_2, 0, x_4] \mid x_2, x_4 \in \mathbb{R} \}$

$= \{ x_2 [0,1,0,0] + x_4 [0,0,0,1] \mid x_2, x_4 \in \mathbb{R} \}$

$\ker(T) = \text{span}\{ [0,1,0,0], [0,0,0,1] \}$

Hence $\{ [0,1,0,0], [0,0,1] \}$ is a basis for $\ker(T)$: Therefore $\dim(\ker(T)) = 2$.

Also, $\text{range}(T) = \{ T([x_1, x_2, x_3, x_4]) \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \}$

$= \{ [x_1, 0, x_3, 0] \mid x_1, x_3 \in \mathbb{R} \}$

$= \{ x_1 [1,0,0,0] + x_3 [0,0,1,0] \mid x_1, x_3 \in \mathbb{R} \}$

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$$\text{range}(T) = \text{Span}\{[1,0,0,0],\ [0,0,1,0]\}$$

Therefore, $\{[1,0,0,0],\ [0,0,1,0]\}$ is a basis for range(T).

Thus, $\dim(\text{range}(T)) = 2$.

Finally, note that

$$\dim(\ker(T)) + \dim(\text{range}(T))$$
$$= 2+2 = 4 = \dim(\mathbb{R}^4).$$

One-to-one and onto linear transformations.

Definition (one-to-one and onto linear transformations)

Let V and W be vector spaces, and let $T: V \to W$ be a linear transformation.

(a) T is said to be one-to-one if distinct vectors in V have different images in W. That is, T is one-to-one if for all $v_1, v_2 \in V$, $T(v_1) = T(v_2)$ implies $v_1 = v_2$.

(b) T is said to be onto if every vectors in W is the images of some vector in V. That is, T is onto if for every $w \in W$, there is some $v \in V$ such that $T(v) = w$.

Example 1) . Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T([x, y, z]) = [2x, x+y+z, -y].$$

Show that T is both one-to-one and onto.

$\rightarrow$ suppose $T([x_1, y_1, z_1]) = T([x_2, y_2, z_2])$

Then, $[2x_1, x_1+y_1+z_1, -z_1] = [2x_2, x_2+y_2+z_2, -z_2]$

$\Rightarrow 2x_1 = 2x_2,\ x_1+y_1+z_1 = x_2+y_2+z_2$ &

$-z_1 = -z_2$

$\Rightarrow x_1 = x_2,\ x_1+y_1+z_1 = x_2+y_2+z_2,$

$z_1 = z_2$

$\Rightarrow x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2.$

$\Rightarrow [x_1, y_1, z_1] = [x_2, y_2, z_2].$

$\therefore$ T is one-to-one.

& Let $[x, y, z] \in \mathbb{R}^3$ be any vector.

Consider $[a, b, c] \in \mathbb{R}^3$ such that

$T([a, b, c]) = [x, y, z].$

$\Rightarrow [2a, a+b+c, -c] = [x, y, z]$

$\Rightarrow$ $a = \frac{1}{2}x$

$c = -z$.

$b = y - \frac{1}{2}x + z$.

Therefore $T\left(\left[\frac{1}{2}x, y-\frac{1}{2}x+z, -z\right]\right) =$

$[x, y, z]$ for each $[x, y, z] \in \mathbb{R}^3$.

Hence T is onto.

(2) Let $T: P_3 \to P_2$ be the linear transformation given by $T(p) = p'$. Show that T is onto but not one-to-one.

$\rightarrow$ Consider the vector $v = x^2+3$ and $v' = x^2+1$, then $v \neq v'$.

Then, $T(v) = 2x$ and $T(v') = 2x$, therefore $T(v) = T(v')$ but $v \neq v'$.

Therefore T is not one-to-one.

Also, T is onto because for every $w = ax^2+bx+c \in P_2$, there is a vector $v = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx + d \in P_3$ such that $T(v) = v' = ax^2+bx+c = w$.

One-to-one linear transformation and kernel.

Theorem :- Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{0_v\}$.

proof :- First we assume that T is one-to-one.

We need to prove that $\ker(T) = \{0_v\}$.

Suppose $v \in \ker(T)$. Then.

$$T(v) = 0_W = T(0_v)$$

$$\Rightarrow T(v) = T(0_v)$$

$$\Rightarrow v = 0_v.$$

Therefore, $\ker(T) \subseteq \{0_v\}$. Clearly, $\{0_v\} \subseteq \ker(T)$.

Hence $\ker(T) = \{0_v\}$.

Conversely, suppose $\ker(T) = \{0_v\}$. We have to show that if $T(v) = T(v')$ then $v = v'$.

Let $T(v) = T(v')$. Then

$$T(v - v') = 0_W$$

$$\Rightarrow v - v' \in \ker(T) = \{0_v\}.$$

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$\Rightarrow v - v' = 0_V$

$\Rightarrow v = 0_V + v' = v'$.

Therefore, T is one-to-one.

Example (1) Consider the linear transformation

$T : P_3 \to P_2$ defined by

$T(ax^3 + bx^2 + cx + d) = ax^2 + bx + c$.

Show that T is onto but not one-to-one.

$\to$ Let $f = ax^3 + bx^2 + cx + d$ be a vector in P_3 such that $T(f) = 0$.

We have
$T(f) = 0$

$\Rightarrow ax^2 + bx + c = 0$

$\Rightarrow a = b = c = 0$.

Therefore $f = d$.

Thus, $\ker(T) \neq \{0\}$.

This implies T is not one-to-one (by previous theorem).

Let $f = ax^2 + bx + c$ be any element in P_2.

Then $T(ax^3 + bx^2 + cx + d) = ax^2 + bx + c$
$= f$.

Therefore, T is onto.

Onto linear transformations and range.

Theorem :- Let V and W be a vector spaces, and let $T: V \to W$ be a linear transformation. If W is finite-dimensional, then T is onto if and only if $\dim(\text{range}(T)) = \dim(W)$.

Corollary :- If $m > n$, then there is no onto linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

proof :- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

By dimension theorem,

$$\dim(\text{range}(T)) = \dim(\mathbb{R}^n) - \dim(\ker(T))$$
$$= n - \dim \ker(T)$$
$$\leq n < m = \dim(\mathbb{R}^m)$$

i.e. $\dim(\text{range}(T)) < \dim(\mathbb{R}^m)$

Thus, by the above theorem, T is not onto.

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Example (1) Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -3 & 4 \\ -6 & 9 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ is one-to-one but not onto.

→ To show that T is one-to-one, it is enough to show that $\ker(T) = \{0\}$.

We now row reduce the matrix A:

$$\begin{bmatrix} -3 & 4 \\ -6 & 9 \\ 7 & -8 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{3}R_1} \begin{bmatrix} 1 & -4/3 \\ -6 & 9 \\ 7 & -8 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 + 6R_1} \begin{bmatrix} 1 & -4/3 \\ 0 & 1 \\ 0 & 4/3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4/3 R_2]{R_1 \to R_1 + 4/3 R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The last matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the reduced row ~~echo~~ echelon form of A having non-zero pivots in columns 1 and 2.

Hence the corresponding columns of matrix A form a basis for range(T), i.e, the set $\{[-3, -6, 7], [4, 9, -8]\}$ forms a basis for range(T), and so dim (range(T)) $= 2$.

By Dimension Theorem, we have

$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(\mathbb{R}^2)$

$\Rightarrow \dim(\ker(T)) + 2 = 2$

$\Rightarrow$

Therefore, $\dim(\ker(T)) = 0$.

$\Rightarrow \ker(T) = \{0\}$.

$\Rightarrow T$ is one-to-one.

(2) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by $T([x, y]) = [x, x+y, x+2y]$. Show that T is one-to-one but not onto.

$\rightarrow$ Let $[x, y] \in \ker(T)$. Then

$T([x, y]) = [0, 0, 0]$

$\Rightarrow [x, x+y, x+2y] = [0, 0, 0]$.

$\Rightarrow x = 0, x+y = 0, x+2y = 0$.

$\Rightarrow x = y = 0$.

$[x, y] \in \{[0, 0]\}$.

$\Rightarrow \therefore \ker T \subseteq \{[0, 0]\}$.

This gives $\ker T = \{[0, 0]\}$

Hence $\dim(\ker T) = 0$.

By Dimension theorem, we have,

$$\dim \mathbb{R}^2 = \dim(\ker T) + \dim(\text{range}(T))$$

$$\Rightarrow 2 = 0 + \dim(\text{range}(T))$$

$$\Rightarrow \dim(\text{range}(T)) = 2 < \dim(\mathbb{R}^3).$$

By previous theorem, T is not onto.