proportional to the first power of the spacing parameter or step size $h$. Since the error depends on the first power of $h$, the formula from ① & (2) are said to provides first order approximation to the first derivatives.

$Q(ii)$. If the error term involves $f^{(n)}$, then the corresponding approximation formula will provides exact result for all polynomial function of degree up to and including $n-1$.

Since the $f^{n}$ is identically zero, so error term vanishes.

(iii) Numerical integration

6.4. The quadrature formula takes the form

$$I(f) \approx I_n(f) = \sum_{i=0}^{n} w_i f(x_i).$$

where $w_i$ = quadrature weights
$x_i$ = quadrature points or abscissas.

→ Newton cotes quadrature rules ~~takes the form~~ is obtain when the $x_i$ are taken as equally spaced point from $[a, b]$ and $w_i$ are computed by fitting and interpolating polynomial to the $f(x_i)$ data and then integrating fn. resulting result.

→ $I(f)$ = exact value of integral of original integrand.
$I_n(f)$ = Newton-Cotes formula.
$I(P_n)$ = Exact value of integral of interpolating polynomial

→ Closed Newton Cotes formula :- It include the end points of the integration interval $x = a$ and $x = b$ among the abscissas.

ie, $\Delta x = \dfrac{(b-a)}{n}$

And the abscissas are $x_i = a + i\Delta x$. for each $i = 0, 1 \ldots n$.

Open- Newton Cotes formula :—

It doesn't include the end pts of the integration interval among the abscissas.

$$\Delta x = \frac{(b-a)}{(n+2)}$$

and the abscissas are $x_i = a + (i+1)\Delta x$ for each $i = 0, 1 \ldots n$

Some Closed Newton Cotes - formula :—

(1) when $n = 1$, This rule is Known as trapezoidal rule's and is given by.

$$I(f) \approx I_{1,\text{closed}}(f) = \frac{\Delta x}{2}[f(a)+f(b)] = \frac{(b-a)}{2}[f(a)+f(b)].$$

and $x = a + t\Delta x$ where $\Delta x = \frac{b-a}{n} = b-a \quad \because n=1$

& $x_0 = a \quad x_1 = b$.

(2). when $n = 2$, This rule is known as, Simpson's rule.

and is given by

$$I(f) \approx I_{2,\text{closed}}(f) = \frac{\Delta x}{3} f(a) + \frac{\Delta x\, 4}{3} f\left(\frac{a+b}{2}\right) + \frac{\Delta x}{3} f(b)$$

$$= \frac{\Delta x}{3}\left[f(a) + 2f(a+b) + f(b)\right].$$

where, $x = a + t\Delta x$.

$x_0 = a \quad x_1 = a + \Delta x \quad x_2 = a + 2\Delta x$.

simplifying w.g.

$x_0 = a \quad x_1 = \frac{a+b}{2}$ & $x_2 = b$.

(3) when $n = 3$, This rule is known as a three-eight rule and is given by

i.e.

$$I(f) \approx I_{3,\text{closed}}(f) = \frac{\Delta x}{8}\left[f(a) + 3f(a+\Delta x) + 3f(a+2\Delta x) + f(b)\right].$$

where $\Delta x = \frac{b-a}{3}$.

(4) when $n = 4$, This rule is known as Booles rule.

$$I(f) \approx I_{4,\text{closed}}(f) = \frac{\Delta x}{90}\Big(7f(a) + 32 f(a+\Delta x) + 12 f(a+2\Delta x) + 32 f(a+3\Delta x) + f(b)\Big).$$

Open Newton Cotes formula :—

(1) when $n=0$, This is called Mid point rule and is given by

$$I(f) \approx I_{0,open}(f) = (b-a) f\left(\frac{a+b}{2}\right)$$ where $\Delta x = \frac{b-a}{2}$ and $x_0 = \frac{a+b}{2}$.

(2) when $n=1$, The ~~gen~~ Newton Cotes is given as,

$$I(f) \approx I_{1,closed}(f) = \frac{\Delta x}{3}\left[f(a+\Delta x) + f(a+2\Delta x)\right].$$

where $\Delta x = \frac{b-a}{3}$, $\left[\because \Delta x = \frac{b-a}{n+2} \text{ for open Newton Cotes}\right]$.

So, $x_0 = a+\Delta x$.
$x_1 = a+2\Delta x$.

(3) when $n=2$, its given by.

$$I(f) \approx I_{2,closed}(f) = \frac{\Delta x}{3}\left[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)\right].$$

where $\Delta x = \frac{b-a}{4}$

(4) when $n=3$, its given by.

$$I(f) \approx I_{3,closed}(f) = \frac{\Delta x}{24}\left[11 f(a+\Delta x) + f(a+2\Delta x) + f(a+3\Delta x) + 11 f(a+4\Delta x)\right].$$

where $\Delta x = \frac{b-a}{5}$

→ Degree of precision of a quadrature rule $I_n(f)$ is the positive integer $m$ s.t.

$I(p) = I_n(p)$ for every polyn of $p^{\text{deg}} \le m$.

$I(p) \neq I_n(p)$ for some polyn. of $p$ of deg $= m+1$.

→ The error term associated with the quadrature rules provides two information.

(I) The error term indicates precisely how the error depends on the length of the integration.

(II). The error terms allows us to integrate the degree of precission, which characterizes the class of polynomial for which the quadrature ~~poro~~ formula produces exact results.

→ If a rule integrates $1, x$ and $x^2$ exactly but fails to integrate $x^3$ exactly, the degree of precission is 2.

→ Weighted Mean Value Theorem :-

Let $f$ be cts on $[a,b]$ and integrable on $[a,b]$ Then $g(x)$ doesn't change sign on $[a,b]$. Then $\exists$ $\xi \in [a,b]$ s.t

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

→ Trapezoidal Error and degree of precission on trapezoidal rule's :-

$$I(f) \approx I_{1,\text{closed}}(f) = -\frac{(b-a)^3}{12} f''(\xi).$$

→ Trapezoidal rule will integrate every constant and every linear polynomial exactly. Since the second derivatives of every constant and every linear polyⁿ is identically zero.

→ Because of the above reason, trapezoidal rule has degree of precission 1.

→ Simpson Error and degree of precission of Simpson Rule :-

1) It has degree of precission 3.

The formula is given by.

$$I(f) \approx I_{2,\text{closed}}(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \quad \text{for } a \le \xi \le b$$

→ Mid point Error and degree of precission of mid point rule :-

It has degree of precision 1.

The formula is given by.

$$I(f) - I_{0,\text{open}}(f) = -\frac{(b-a)^3}{24} f(\xi) \quad \text{for } a \le \xi \le b$$

and $\int_a^b f(x)\,dx = (b-a) f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f^{(11)}(\xi) \quad a \le \xi \le b$