62. Numerical Differentiation.

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Let "f" be an arbitrary function. If "f" is interpolated through $x=x_0$ and $x=x_1$, interpolation theory guarantees that

$$f(x) = \underbrace{\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)}_{\text{Lagrange interpolating poly}} + \underbrace{f[x_0,x_1,x](x-x_0)(x-x_1)}_{\text{Error term.}} \quad \text{——— (1)}$$

differentiate f with respect to x, we obtain

$$f'(x) = \frac{1}{x_0-x_1} f(x_0) + \frac{1}{x_1-x_0} f(x_1) + (x-x_0)(x-x_1)\frac{d}{dx} f[x_0,x_1,x]$$
$$+ f[x_0,x_1,x](2x-x_0-x_1)$$

Evaluating this expression at $x=x_0$ then yields, after some simplification,

$$f'(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} + f[x_0,x_1,x_0](x_0-x_1)$$

if "f" has two cts derivatives, then there exists a ξ between x_0 and x_1 such that

$$f[x_0,x_1,x_0] = \frac{f''(\xi)}{2}$$

Therefore

$$f'(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} + \frac{x_0-x_1}{2} f''(\xi)$$

or

$$\boxed{f'(x_0) \approx \frac{f(x_1)-f(x_0)}{x_1-x_0}} \quad \text{——— (2)}$$

Since data points are equally spaced i.e $x_1-x_0=h$
or $x_1=x_0+h$

Where h is step length.

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from eq^n (2) we get

$$\boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi)}$$

or
$$\boxed{f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}}$$
where $x_0 < \xi < x_0+h$

This is forward difference approximation for first derivative.

where $x_0 < \xi < x_0+h$, while substituting $x_1 = x_0 - h$ into eq^n (2) produces

$$\boxed{f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} + \frac{h}{2} f''(\xi)}$$

or
$$\boxed{f'(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h}}$$

This is Known as Backward difference approximation to first derivate.

Example 6.1: Find Approximating the derivate of the Natural Logarithm. (Forward and Backward)

Consider the function $f(x) = \ln x$.

$$f'(x) = \frac{1}{x}$$

$$\text{at } x_0 = 2, \ f'(x_0) = f'(2) = \frac{1}{2} = 0.5$$

Exact value of $f'(x)$ at $x_0 = 2$ is 0.5.

Now, we are finding approximate value of $f'(x)$ at $x_0 = 2$, by def of forward and backward difference

Taking $h=1$, $x_0=2$ (given)

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$$

$$\approx \frac{f(2+1)-f(2)}{1}$$

$$\approx \ln 3 - \ln 2 = 0.405465$$

Error = True value − approximate value

$= 0.5 - 0.405465 = 0.094535$

Taking $\underline{h=0.1}$.

$$f'(2) \approx \frac{\ln(2.1)-\ln(2.0)}{0.1} = 0.487902$$

Error $= 0.5 - 0.487902 = 0.012098$

Taking $\underline{h=0.01}$

$$f'(2) \approx \frac{\ln(2.01)-\ln(2.0)}{0.01} = 0.498754$$

Error $= 0.5 - 0.498754$

$= 1.2458 \times 10^{-3}$

Taking $\underline{h=0.001}$

$$f'(2) \approx \frac{\ln(2.001)-\ln(2.0)}{0.001} = 0.499875$$

Error $= 0.5 - 499875 = 1.2496 \times 10^{-4}$

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Similarly. By Backward difference approximation method

h	$\frac{f(x_0)-f(x_0-h)}{h}$	error
1.0	$f'(2) \approx \frac{\ln(2)-\ln(1.0)}{1.0} = 0.693147$	0.193147
0.1	$f'(2) \approx \frac{\ln(2)-\ln(1.9)}{0.1} = 0.512933$	0.012933
0.01	$f'(2) \approx \frac{\ln(2)-\ln(1.99)}{0.01} = 0.501254$	1.2542×10^{-3}
0.001	$f'(2) \approx \frac{\ln(2)-\ln(1.999)}{0.001} = 0.500125$	1.2505×10^{-4}

Note: At each time the step size, h, is cut by a factor $1/10$, the corresponding error also drops by a factor of 10. its means that approx. solⁿ converges to exact solⁿ, if $h\to0$. (size of step size is very small as $h\to0$).

Example 6.2: To verify that Forward and Backward difference approximation to first derivative are exact for constant and linear function.

We consider two fⁿs $f(x)=1$ and $f(x)=x$

	$f'(x_0)$	$\frac{f(x_0+h)-f(x_0)}{h}$	$\frac{f(x_0)-f(x_0-h)}{h}$
$f(x)=1$	0	0	0
$f(x)=x$	1	1	1
$f(x)=x^2$	$2x_0$	$2x_0+h$	$2x_0-h$

Hence forward and Backward difference approx. to first derivative are exact for constant and linear functions

Q.1 . We have to obtain Central difference approximation . for first order derivate.

Note : All students define and send to me .

Q.2 : Do. **Example 6.3** :

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