Lecture : 3

(APPLICATIONS OF LAPLACE TRANSFORMS)

Course : B.Sc. (H) Physics
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Topics covered in this lecture:-

- Applications of LT to Second order Differential equations, Coupled Differential equations & solution of heat flow along semi-infinite bar

3.1 Ordinary Differential equations with constant coefficients:

Ordinary Differential equations with constant coefficients can be very easily solved using Laplace transform without finding the general solution and the arbitrary constants.

Examples:

1. Solve $y'' - 2y' + 2y = 0$, given $y = y' = 1$ when $t = 0$.

Sol. We have, $y'' - 2y' + 2y = 0$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\{y''\} - 2L\{y'\} + 2L\{y\} = L\{0\}$$

$$\Rightarrow \quad [s^2 L\{y\} - s\,y(0) - y'(0)] - 2[s\,L\{y\} - y(0)] + 2L\{y\} = 0$$

$$[\because L\{f'(t)\} = sF(s) - f(0) \text{ and } L\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)]$$

$$\Rightarrow \quad (s^2 - 2s + 2)\,L\{y\} - (s-2)\,1 - (1) = 0 \qquad [\because y(0) = y'(0) = 1]$$

$$\Rightarrow \quad L\{y\} = \frac{s-1}{s^2 - 2s + 2} = \frac{s-1}{(s-1)^2 + 1} \qquad (2)$$

Taking Inverse Laplace transform of both sides of eq. (2), we get

$$y = L^{-1}\left\{\frac{s-1}{(s-1)^2 + 1}\right\}$$

$$= e^t L^{-1}\left\{\frac{s}{s^2 + 1^2}\right\} = e^t \cos t \text{, the required solution.}$$

2. Using Laplace Transform method, solve $\frac{d^2y}{dt^2} + y = t$, given $\frac{d^2y}{dt^2} = 1$, when $t = 0$ and $y = 0$ when $t = \pi$.

Sol. We have, $\frac{d^2y}{dt^2} + y = t$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{d^2y}{dt^2}\right\} + L\{y\} = L\{t\}$$

$$\Rightarrow \quad [s^2 L\{y\} - s\, y(0) - y'(0)] + L\{y\} = \frac{1}{s^2}$$

$$[\because L\{f'(t)\} = sF(s) - f(0) \text{ and } L\{f''(t)\} = s^2 F(s) - s\, f(0) - f'(0)]$$

$$\Rightarrow \quad (s^2+1)\, L\{y\} - s\, y(0) - y'(0) = \frac{1}{s^2} \qquad (2)$$

Now, let at $t = 0$, $y(0) = a$ and $y'(0) = 1$ (given) (3)

Put eq. (3) in eq. (2), we get

$$\Rightarrow \quad (s^2+1)\, L\{y\} - s\, a - 1 = \frac{1}{s^2}$$

$$\Rightarrow \quad L\{y\} = \frac{s\,a}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} \qquad (4)$$

Taking Inverse Laplace transform of both sides of eq. (4), we get

$$y = a\, L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$$

$$\Rightarrow \quad y = a\cos t + \sin t + L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} \qquad (5)$$

Now, $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$

$$= L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$= t - \sin t \qquad (6)$$

Put eq. (6) in eq. (5), we get

$$y = a\cos t + \sin t + t - \sin t = t + a\cos t \qquad (7)$$

Now, $y = 0$ when $t = \pi$

$\therefore \quad 0 = \pi + a\cos\pi$

$\Rightarrow \quad 0 = \pi + a(-1) \qquad \Rightarrow \qquad a = \pi$

Put $a = \pi$ in eq. (7), we get

$\therefore \quad y = t + \pi\cos t$, the required solution.

3. Solve $y''' - 3y'' + 3y' - y = t^2 e^t$, given $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$.

Sol. We have, $y''' - 3y'' + 3y' - y = t^2 e^t$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\Rightarrow \quad [s^3 L\{y\} - s^2 y(0) - s\, y'(0) - y''(0)] - 3[s^2 L\{y\} - s\, y(0) - y'(0)] + 3[s\, L\{y\} - y(0)] - L\{y\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

$[\because L\{f'(t)\} = sF(s) - f(0)$, $L\{f''(t)\} = s^2 F(s) - s\, f(0) - f'(0)$,

$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s\, f'(0) - f''(0)$, and

$$L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}]$$

$\Rightarrow$ $[s^3 L\{y\} - s^2(1) - s(0) - (-2)] - 3[s^2 L\{y\} - s(1) - (0)] + 3[s L\{y\} - (1)]$

$$- L\{y\} = \frac{2}{(s-1)^3} \qquad [\because\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2]$$

$\Rightarrow$ $(s^3 - 3s^2 + 3s - 1)\, L\{y\} - s^2 + 3s - 1 = \dfrac{2}{(s-1)^3}$

$\Rightarrow$ $(s^3 - 3s^2 + 3s - 1)\, L\{y\} = \dfrac{2}{(s-1)^3} + s^2 - 3s + 1$

$\Rightarrow$ $L\{y\} = \dfrac{2}{(s-1)^3(s^3 - 3s^2 + 3s - 1)} + \dfrac{s^2 - 3s + 1}{(s^3 - 3s^2 + 3s - 1)}$

$\Rightarrow$ $L\{y\} = \dfrac{2}{(s-1)^3 (s-1)^3} + \dfrac{s^2 - 3s + 1}{(s-1)^3}$

$\Rightarrow$ $y = L^{-1}\left\{\dfrac{2}{(s-1)^6}\right\} + L^{-1}\left\{\dfrac{(s-1)^2 - (s-1) - 1}{(s-1)^3}\right\}$

$\Rightarrow$ $y = e^t L^{-1}\left\{\dfrac{2}{s^6}\right\} + e^t L^{-1}\left\{\dfrac{s^2 - s - 1}{s^3}\right\}$ $\qquad [\because\ L^{-1}\{F(s-b)\} = e^{bt} L^{-1}\{F(s)\}]$

$\Rightarrow$ $y = 2e^t L^{-1}\left\{\dfrac{1}{s^6}\right\} + e^t L^{-1}\left\{\dfrac{1}{s} - \dfrac{1}{s^2} - \dfrac{1}{s^3}\right\}$

$\Rightarrow$ $y = 2e^t \dfrac{t^5}{5!} + e^t\left[1 - t - \dfrac{t^2}{2}\right]$ $\qquad \left[\because\ L^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}\right]$

$\Rightarrow$ $y = e^t\left[1 - t - \dfrac{t^2}{2} + \dfrac{t^5}{60}\right]$, the required solution.

3.2 Ordinary Differential equations with variable coefficients:

Ordinary Differential equations with variable coefficients can be very easily solved using Laplace transform.

Examples:

1. Using Laplace Transform, solve the following differential equation

$$y'' + 2t\, y' - y = t, \quad \text{when} \quad y(0) = 0,\ y'(0) = 1$$

Sol. We have, $y'' + 2t\, y' - y = t$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\{y''\} + 2L\{t\, y'\} - L\{y\} = L\{t\}$$

$\Rightarrow$ $[s^2 L\{y\} - s\, y(0) - y'(0)] - 2\dfrac{d}{ds}[s\, L\{y\} - y(0)] - L\{y\} = \dfrac{1}{s^2}$

$$[\because\ L\{f'(t)\} = sF(s) - f(0)\ ,\ L\{f''(t)\} = s^2 F(s) - s\, f(0) - f'(0)]$$

$\Rightarrow$ $[s^2 L\{y\} - s(0) - (1)] - 2\dfrac{d}{ds}[s\, L\{y\} - 0] - L\{y\} = \dfrac{1}{s^2}$ $\qquad [\because\ y(0) = 0,\ y'(0) = 1]$

$$\Rightarrow \quad [s^2 L\{y\} - 1] - 2\frac{d}{ds}[s\, L\{y\}] - L\{y\} = \frac{1}{s^2}$$

$$\Rightarrow \quad [s^2 L\{y\} - 1] - 2L\{y\}] - L\{y\} = \frac{1}{s^2}$$

$$\Rightarrow \quad (s^2 - 3)\, L\{y\} - 1 = \frac{1}{s^2}$$

$$\Rightarrow \quad (s^2 - 3)\, L\{y\} = \frac{1}{s^2} + 1 = \frac{s^2 + 1}{s^2}$$

$$\Rightarrow \quad L\{y\} = \frac{s^2 + 1}{s^2\,(s^2 - 3)}$$

$$= \frac{s^2}{s^2\,(s^2 - 3)} + \frac{1}{s^2\,(s^2 - 3)}$$

$$= \frac{1}{s^2 - 3} + \frac{1}{3}\left(\frac{1}{s^2 - 3} - \frac{1}{s^2}\right)$$

$$= \frac{4}{3}\,\frac{1}{s^2 - 3} - \frac{1}{3s^2} \qquad (2)$$

Taking Inverse Laplace transform of both sides of eq. (2), we get

$$y = \frac{4}{3}\, L^{-1}\left\{\frac{1}{s^2 - (\sqrt{3})^2}\right\} - \frac{1}{3}\, L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\Rightarrow \quad y = \frac{4}{3}\,\frac{1}{\sqrt{3}}\, L^{-1}\left\{\frac{\sqrt{3}}{s^2 - (\sqrt{3})^2}\right\} - \frac{1}{3}\, t$$

$$\Rightarrow \quad y = \frac{4}{3}\,\frac{1}{\sqrt{3}} \sinh \sqrt{3}\, t - \frac{1}{3}\, t$$, the required solution.

2. A particle moves in a line so that its displacement x from a fixed point O at any time t, is given by

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t$$

Using Laplace transform, find its displacement at any time t if initially particle is at rest at $x = 0$.

Sol. We have,

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t, \qquad x(0) = 0,\ x'(0) = 0 \qquad (1)$$

Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{d^2x}{dt^2}\right\} + 4L\left\{\frac{dx}{dt}\right\} + 5L\{x\} = 80\, L\{\sin 5t\}$$

$$\Rightarrow \quad [s^2 L\{x\} - s\, x(0) - x'(0)] + 4[s\, L\{x\} - x(0)] + 5L\{x\} = 80\left(\frac{5}{s^2 + 25}\right)$$

$$[\because L\{f'(t)\} = sF(s) - f(0),\ L\{f''(t)\} = s^2 F(s) - s\, f(0) - f'(0)]$$

$$\Rightarrow \quad [s^2 L\{x\} - s(0) - 0)] + 4[s L\{x\} - 0] + 5L\{x\} = \frac{400}{s^2+25}$$

$[\because \ x(0) = 0, \ x'(0) = 0]$

$$\Rightarrow \quad (s^2+4s+5)\, L\{x\} = \frac{400}{s^2+25}$$

$$\Rightarrow \quad L\{x\} = \frac{400}{(s^2+4s+5)(s^2+25)} \qquad (2)$$

Taking Inverse Laplace transform of both sides of eq. (2), we get

$$x = L^{-1}\left\{\frac{400}{(s^2+4s+5)(s^2+25)}\right\}$$

$$\Rightarrow \quad x = 400\,L^{-1}\left\{\frac{As+B}{s^2+4s+5} + \frac{Cs+D}{s^2+25}\right\}$$

$$\Rightarrow \quad x = 400A\; L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+1}\right\} + 400B\; L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} + 400C\; L^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{400}{5} D\; L^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$\Rightarrow \quad x = 400A\, e^{-2t}\, L^{-1}\left\{\frac{s-2}{s^2+1}\right\} + 400B\, e^{-2t}\, L^{-1}\left\{\frac{1}{s^2+1}\right\} + 400C \cos 5t + 80D \sin 5t$$

$$\Rightarrow \quad x = 400A\, e^{-2t}[\cos t - 2\sin t] + 400B\, e^{-2t} \sin t + 400C \cos 5t + 80D \sin 5t$$

To find constants A, B, C, D:

$$\frac{1}{(s^2+4s+5)(s^2+25)} = \frac{As+B}{s^2+4s+5} + \frac{Cs+D}{s^2+25}$$

$$\Rightarrow \quad 1 = (As+B)(s^2+25) + (Cs+D)(s^2+4s+5)$$

s^3: $\quad 0 = A + C \quad \Rightarrow \quad C = -A$

s^2: $\quad 0 = B + 4C + D \quad \Rightarrow \quad B + D = 4A$

s: $\quad 0 = 25A + 5C + 4D \quad \Rightarrow \quad D = -5A \quad \Rightarrow \quad B = 9A$

s^0: $\quad 1 = 25B + 5D \quad \Rightarrow \quad 1 = 225A - 25A \quad \Rightarrow \quad A = 1/200$

$\therefore \quad B = 9/200, \ C = -1/200, \ D = -5/200$

Put these values in x, we get

$$x = 2e^{-2t}[\cos t - 2\sin t] + 18e^{-2t} \sin t - 2\cos 5t - 2\sin 5t$$

$\Rightarrow \quad x = 2e^{-2t}[\cos t + 7\sin t] - 2(\cos 5t + \sin 5t)$, the required displacement.

3.3 Solution of Simultaneous Ordinary Differential equations:

Simultaneous Ordinary Differential equations can also be solved using Laplace transform.

Examples:

1. Solve $\begin{cases} \dfrac{dx}{dt} = 2x - 3y \\ \dfrac{dy}{dt} = y - 2x \end{cases}$ subject to $x(0) = 8, \ y(0) = 3$

Sol. We have to solve

$$x' = 2x - 3y$$

and $$y' = y - 2x$$

Taking Laplace transform of both sides, we get

$$L\{x'\} = 2L\{x\} - 3L\{y\}$$

and $$L\{y'\} = L\{y\} - 2L\{x\}$$

$\Rightarrow$ $$sL\{x\} - x(0) = 2L\{x\} - 3L\{y\}$$

and $$sL\{y\} - y(0) = L\{y\} - 2L\{x\} \qquad [\because L\{f'(t)\} = sF(s) - f(0)]$$

$\Rightarrow$ $$sL\{x\} - 8 = 2L\{x\} - 3L\{y\}$$

and $$sL\{y\} - 3 = L\{y\} - 2L\{x\} \qquad [\because x(0) = 8,\ y(0) = 3]$$

$\Rightarrow$ $$(s-2)\,L\{x\} + 3L\{y\} = 8$$

and $$2L\{x\} + (s-1)\,L\{y\} = 3$$

Using Cramer's rule, we get

$$L\{x\} = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8(s-1)-9}{(s-2)(s-1)-6} = \frac{8s-8-9}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} \qquad (1)$$

and $$L\{y\} = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3(s-2)-16}{(s-2)(s-1)-6} = \frac{3s-6-16}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} \qquad (2)$$

Taking Inverse Laplace transform of both sides of eqs. (1) and (2), we get

$$x = L^{-1}\left\{\frac{8s-17}{(s+1)(s-4)}\right\} = L^{-1}\left\{\frac{A}{s+1} + \frac{B}{s-4}\right\}$$

$$= A\,L^{-1}\left\{\frac{1}{s+1}\right\} + B\,L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= A\,e^{-t} + B\,e^{4t} \qquad (3)$$

and $$y = L^{-1}\left\{\frac{3s-22}{(s+1)(s-4)}\right\} = L^{-1}\left\{\frac{C}{s+1} + \frac{D}{s-4}\right\}$$

$$= C\,L^{-1}\left\{\frac{1}{s+1}\right\} - D\,L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= C\,e^{-t} - D\,e^{4t} \qquad (4)$$

To find constants A and B:

$$\frac{8s-17}{(s+1)(s-4)} = \frac{A}{s+1} + \frac{B}{s-4}$$

$\Rightarrow \quad 8s - 17 = A\,(s-4) + B\,(s+1)$

Put $s = 4$: $\quad 15 = 5B \quad \Rightarrow \quad B = 3$

Put $s = -1$: $\quad -25 = -5A \quad \Rightarrow \quad A = 5$

To find constants C and D:

$$\frac{3s-22}{(s+1)\,(s-4)} = \frac{C}{s+1} - \frac{D}{s-4}$$

$\Rightarrow \quad 3s - 22 = C\,(s-4) + D\,(s+1)$

Put $s = 4$: $\quad -10 = 5D \quad \Rightarrow \quad D = -2$

Put $s = -1$: $\quad -25 = -5C \quad \Rightarrow \quad C = 5$

Put the values of these constants in eqs. (3) and (4), we get

$$x = 5e^{-t} + 3e^{4t}$$

and $\quad y = 5e^{-t} - 2e^{4t}$, the required solution.

2. Solve $\begin{cases} x' + x + y = 0 \\ y' + 4x + y = 0 \end{cases}$ subject to $x(0) = y(0) = 1$, where $x' = \dfrac{dx}{dt}$, $y' = \dfrac{dy}{dt}$

Sol. We have,

$$x' + x + y = 0$$

and $\quad y' + 4x + y = 0$

Taking Laplace transform of both sides, we get

$$L\{x'\} + L\{x\} + L\{y\} = L\{0\}$$

and $\quad L\{y'\} + 4L\{x\} + L\{y\} = L\{0\}$

$\Rightarrow \quad s\,L\{x\} - x(0) + L\{x\} + L\{y\} = 0$

and $\quad s\,L\{y\} - y(0) + 4L\{x\} + L\{y\} = 0 \qquad [\because L\{f'(t)\} = sF(s) - f(0)]$

$\Rightarrow \quad s\,L\{x\} - 1 + L\{x\} + L\{y\} = 0$

and $\quad s\,L\{y\} - 1 + 4L\{x\} + L\{y\} = 0 \qquad [\because x(0) = y(0) = 1]$

$\Rightarrow \quad (s+1)\,L\{x\} + L\{y\} = 1$

and $\quad 4L\{x\} + (s+1)\,L\{y\} = 1$

Using Cramer's rule, we get

$$L\{x\} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & s+1 \end{vmatrix}}{\begin{vmatrix} s+1 & 1 \\ 4 & s+1 \end{vmatrix}} = \frac{s}{(s+1)^2 - 4} = \frac{s}{s^2 + 2s - 3} = \frac{s}{(s+3)\,(s-1)} \qquad (1)$$

and
$$L\{y\} = \frac{\begin{vmatrix} s+1 & 1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} s+1 & 1 \\ 4 & s+1 \end{vmatrix}} = \frac{s+1-4}{(s+1)^2 - 4} = \frac{s-3}{s^2 + 2s - 3} = \frac{s-3}{(s+3)\,(s-1)} \qquad (2)$$

Taking Inverse Laplace transform of both sides of eqs. (1) and (2), we get

$$x = L^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\} = L^{-1}\left\{\frac{A}{s+3} + \frac{B}{s-1}\right\}$$

$$= A\, L^{-1}\left\{\frac{1}{s+3}\right\} + B\, L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= A\, e^{-3t} + B\, e^{t} \qquad (3)$$

and $$y = L^{-1}\left\{\frac{s-3}{(s+3)(s-1)}\right\} = L^{-1}\left\{\frac{C}{s+3} + \frac{D}{s-1}\right\}$$

$$= C\, L^{-1}\left\{\frac{1}{s+3}\right\} + D\, L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= C\, e^{-3t} + D\, e^{t} \qquad (4)$$

To find constants A and B:

$$\frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$\Rightarrow \quad s = A\,(s-1) + B\,(s+3)$

Put $s = 1$: $\quad 1 = 4B \quad \Rightarrow \quad B = 1/4$

Put $s = -3$: $\quad -3 = -4A \quad \Rightarrow \quad A = 3/4$

To find constants C and D:

$$\frac{s-3}{(s+3)(s-1)} = \frac{C}{s+3} + \frac{D}{s-1}$$

$\Rightarrow \quad s - 3 = C\,(s-1) + D\,(s+3)$

Put $s = 1$: $\quad -2 = 4D \quad \Rightarrow \quad D = -1/2$

Put $s = -3$: $\quad -6 = -4C \quad \Rightarrow \quad C = 3/2$

Put the values of these constants in eqs. (3) and (4), we get

$$x = \frac{3}{4}\, e^{-3t} + \frac{1}{4}\, e^{t}$$

and $y = \dfrac{3}{2}\, e^{-3t} - \dfrac{1}{2}\, e^{t}$, the required solution.

3.4 Solution of Partial Differential equations:

Given the function $u(x, t)$ defined for $a \le x \le b$, $t > 0$,then

(a) $$L\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty e^{-st}\, \frac{\partial u}{\partial t}\, dt = s\, U(x, s) - u(x, 0),$$

(b) $$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\, \frac{\partial u}{\partial x}\, dt = \frac{dU}{dx},$$

(c) $$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\, U(x, s) - s\, u(x, 0) - u_t(x, 0),$$

(d) $$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 U}{dx^2}$$

where, $u_t(x, s) = \left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ and $U = U(x, s) = L\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t)\, dt$

Proof:

(a) $$L\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty e^{-st} \frac{\partial u}{\partial t}\, dt$$

$$= e^{-st} u(x, t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} u(x, t)\, dt \} \qquad \text{[Integrating by parts]}$$

$$= 0 - u(x, 0) + s\int_0^\infty e^{-st} u(x, t)\, dt$$

$$= s\, U(x, s) - u(x, 0)$$

$$= s\, U - u(x, 0)$$

(b) $$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial u}{\partial x}\, dt$$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-st} u\, dt = \frac{\partial U}{\partial x}$$

$$= \frac{dU}{dx}$$

(c) Let $v = \dfrac{\partial u}{\partial t}$

then $$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = s\, L\{v\} - v(x, 0)$$

$$= s\, L\{\partial u / \partial t\} - v(x, 0)$$

$$= s[s\, L\{u\} - u(x, 0)] - u_t(x, 0)$$

$$= s^2 L\{u\} - s\, u(x, 0) - u_t(x, 0)$$

(d) Let $w = \dfrac{\partial u}{\partial x}$

then $$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = L\left\{\frac{\partial w}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial w}{\partial x}\, dt$$

$$= \int_0^\infty e^{-st} \frac{\partial^2 u}{\partial x^2}\, dt$$

$$= \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} u\, dt = \frac{\partial^2 U}{\partial x^2}$$

$$= \frac{d^2 U}{dx^2}$$

Examples:

1. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $u(x, 0) = 3\sin 2\pi x$, $u(0, t) = 0$, $u(1, t) = 0$ where $0 < x < 1$, $t > 0$.

Sol. We have, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad s\,U - u(x, 0) = \frac{d^2U}{dx^2}$$

$$\left[\because L\left\{\frac{\partial u}{\partial t}\right\} = s\,U - u(x, 0),\ L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2} \text{ and } U = U(x, s) = L\{u(x, t)\}\right]$$

$$\Rightarrow \quad s\,U - 3\sin 2\pi x = \frac{d^2U}{dx^2} \qquad [\because u(x, 0) = 3\sin 2\pi x]$$

$$\Rightarrow \quad \frac{d^2U}{dx^2} - s\,U = 3\sin 2\pi x \tag{2}$$

The auxiliary equation: $(m^2 - s) = 0 \Rightarrow m = \pm\sqrt{s}$

Complimentary function, $U_c = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}}$

Particular solution, $U_p = \dfrac{-3}{D^2 - s}\sin 2\pi x = \dfrac{-3}{-(2\pi)^2 - s}\sin 2\pi x$

$$= \frac{3\sin 2\pi x}{4\pi^2 + s}$$

Thus, the solution of eq. (2) is

$$U = U_c + U_p = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}} + \frac{3\sin 2\pi x}{4\pi^2 + s} \tag{3}$$

Now we have $u(0, t) = 0$ and $u(1, t) = 0$

Taking Laplace transform of both sides, we get

$L\{u(0, t)\} = L\{0\}$ and $L\{u(1, t)\} = L\{0\}$

$\Rightarrow \quad U(0, s) = 0$ and $U(1, s) = 0$ (4), (5)

Using eqs. (4), (5) in eq. (3), we get

$0 = C_1 + C_2$ and $0 = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}}$

$\Rightarrow \quad C_1 = C_2 = 0$ (6)

Put eq. (6) in eq. (3), we get

$$U = \frac{3\sin 2\pi x}{4\pi^2 + s} \tag{7}$$

Taking Inverse Laplace transform of both sides of eq. (7), we get

$$u = L^{-1}\{U\} = L^{-1}\left\{\frac{3\sin 2\pi x}{s + 4\pi^2}\right\}$$

$$= 3\sin(2\pi x)\, L^{-1}\left\{\frac{1}{s + 4\pi^2}\right\}$$

$= 3 \sin(2\pi x)\, e^{-4\pi^2 t}$, the required solution.

2. Find the solution of $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x, 0) = 6e^{-3x}$ which is bounded for $x > 0,\ t > 0$.

Sol. We have, $\frac{\partial u(x,t)}{\partial x} = 2\frac{\partial u(x,t)}{\partial t} + u(x,t)$ (1)

Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{\partial u(x,t)}{\partial x}\right\} = 2L\left\{\frac{\partial u(x,t)}{\partial t}\right\} + L\{u(x,t)\}$$

$$\Rightarrow \quad \frac{dU}{dx} = 2[sU - u(x,0)] + U$$

$$\left[\because\ L\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{dU}{dx} \text{ and } L\left\{\frac{\partial u(x,t)}{\partial t}\right\} = sU - u(x,0)\right]$$

$$\Rightarrow \quad \frac{dU}{dx} = 2[sU - 6e^{-3x}] + U \qquad [\because\ u(x,0) = 6e^{-3x}]$$

$$\Rightarrow \quad \frac{dU}{dx} - 2sU = -12e^{-3x} \tag{2}$$

To solve eq. (2), let us find the integrating factor, $IF = e^{\int -(2s+1)dx} = e^{-(2s+1)x}$

$\therefore$ eq. (2) becomes:

$$U(IF) = \int (IF)(-12)\, e^{-3x}\, dx$$

$$\Rightarrow \quad U e^{-(2s+1)x} = -12\int e^{-(2s+1)x} e^{-3x}\, dx$$

$$= -12\int e^{-2(s+2)x}\, dx$$

$$= \frac{-12e^{-2(s+2)x}}{-2(s+2)} + \mathrm{C}, \text{ C is the constant of integration.}$$

$$= \frac{6\, e^{-2(s+2)x}}{s+2} + \mathrm{C}$$

$$\Rightarrow \quad U = \frac{6}{s+2} e^{-3x} + \mathrm{C}e^{(2s+1)x} \tag{3}$$

Now, since $u(x,t)$ must be bounded as $x \to \infty$, we must have $U(x,s)$ also bounded as $x \to \infty$ and it follows that we must choose C = 0.

so, eq. (3) becomes:

$$U = \frac{6}{s+2} e^{-3x} \tag{4}$$

Taking Inverse Laplace transform of both sides of eq. (6), we get

$$u = L^{-1}\{U\} = L^{-1}\left\{\frac{6}{s+2} e^{-3x}\right\}$$

$$= e^{-3x} L^{-1}\left\{\frac{6}{s+2}\right\}$$

$$= e^{-3x} e^{-2t}$$

$= e^{-(3x+2t)}$, the required solution.

3.5 Solution of semi-infinite bar using Laplace transform:

1. A semi-infinite solid $x > 0$ is initially at temperature zero. At $t = 0$, a constant temperature $u_o > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at any later time $t > 0$.

Sol. The boundary-value problem for the determination of the temperature $u(x, t)$ at any point x and any time t is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad x > 0,\ t > 0 \qquad (1)$$

s. t. $u(x, 0) = 0, \qquad u(0, t) = u_o, \qquad |u(x, t)| < M \qquad (2)$

where the last condition expresses the requirement that the temperature is bounded $\forall\ x$ and t.
Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = k\ L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad s\,U - u(x, 0) = k\,\frac{d^2U}{dx^2} \qquad \left[\because\ L\left\{\frac{\partial u}{\partial t}\right\} = s\,U - u(x, 0) \text{ and } L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2}\right]$$

$$\Rightarrow \quad s\,U - 0 = k\,\frac{d^2U}{dx^2} \qquad [\because\ u(x, 0) = 0 \text{ from eq. (2)}]$$

$$\Rightarrow \quad \frac{d^2U}{dx^2} - \frac{s}{k}\,U = 0 \qquad (4)$$

Now to get the solution of eq. (4):

The auxiliary equation is $\left(m^2 - \frac{s}{k}\right) = 0 \quad \Rightarrow \quad m = \pm\sqrt{\frac{s}{k}}$

$\therefore$ The solution of eq. (4) is

$$U(x, s) = C_1\, e^{x\sqrt{s/k}} + C_2\, e^{-x\sqrt{s/k}} \qquad (5)$$

We choose $C_1 = 0$ so that $U(x, s)$ is bounded as $x \to \infty$, and we get

$$U(x, s) = C_2\, e^{-x\sqrt{s/k}} \qquad (6)$$

Also from eq. (2) we have $\quad u(0, t) = u_o$

Taking Laplace transform of both sides of the above equation, we get

$$U(0, s) = L\{u(0, t)\} = L\{u_o\} = u_o\, L\{1\} = \frac{u_o}{s} \qquad (7)$$

Put eq. (7) in eq. (6), we get

$$U(0, s) = C_2\,(1) = \frac{u_o}{s} \qquad \Rightarrow \qquad C_2 = \frac{u_o}{s} \qquad (8)$$

Put eq. (8) in eq. (6), we get

$$U(x, s) = \frac{u_o}{s}\, e^{-x\sqrt{s/k}} \qquad (9)$$

Taking Inverse Laplace transform of both sides of eq. (9), we get

$$u(x, t) = L^{-1}\{U\} = L^{-1}\left\{\frac{u_o}{s}\, e^{-x\sqrt{s/k}}\right\} = u_o\, L^{-1}\left\{\frac{e^{-x\sqrt{s/k}}}{s}\right\}$$

$$= u_o\ erfc\ (x/2\sqrt{kt}\ ) \qquad \left[\because\ L^{-1}\left\{\frac{e^{-x\sqrt{s/k}}}{s}\right\} = erfc\left(x/2\sqrt{kt}\right)\right]$$

the required temperature.

2. A semi-infinite insulated bar which coincides with the x axis, $x > 0$, is initially at temperature zero. At $t = 0$, a quantity of heat is instantaneously generated at the point $x = a$ where $a > 0$. Find the temperature at any point of the bar at any time $t > 0$.

Sol. The equation for heat conduction in the bar is

$$\frac{\partial u}{\partial t} = k\ \frac{\partial^2 u}{\partial x^2} \qquad x > 0,\ t > 0 \qquad (1)$$

The fact that a quantity of heat is instantaneously generated at the point $x = a$ can be represented by the boundary condition

$$u(a,\ t) = q\ \delta(t) \qquad (2)$$

where q is a constant and $\delta(t)$ is the Dirac delta function.

Also, since the initial temperature is zero and since the temperature must be bounded, we have

$$u(x,\ 0) = 0, \qquad |u(x,\ t)| < M \qquad (3)$$

Taking Laplace transform of both sides of eq. (1), we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = k\ L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad s\ U - u(x,\ 0) = k\ \frac{d^2U}{dx^2} \qquad [\because\ \ L\left\{\frac{\partial u}{\partial t}\right\} = s\ U - u(x,\ 0) \text{ and } L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2}\ ]$$

$$\Rightarrow \quad s\ U - 0 = k\ \frac{d^2U}{dx^2} \qquad [\because\ \ u(x,\ 0) = 0]$$

$$\Rightarrow \quad \frac{d^2U}{dx^2} - \frac{s}{k}\ U = 0 \qquad (4)$$

Now, $u(a,\ t) = q\ \delta(t)$

Taking Laplace transform of both sides, we get

$$L\{u(a,\ t)\} = L\{q\ \delta(t)\}$$

$$\Rightarrow \quad U(a,\ s) = q\ L\{\delta(t)\} = q \qquad (5)$$

Now to get the solution of eq. (4):

The auxiliary equation is $\left(m^2 - \frac{s}{k}\right) = 0 \qquad \Rightarrow \qquad m = \pm\sqrt{\frac{s}{k}}$

$\therefore$ The solution of eq. (4) is

$$U(x,\ s) = C_1\ e^{x\sqrt{s/k}} + C_2\ e^{-x\sqrt{s/k}} \qquad (6)$$

We choose $C_1 = 0$ so that $U(x,\ s)$ is bounded as $x \to \infty$, and we get

$$U(x,\ s) = C_2\ e^{-x\sqrt{s/k}} \qquad (7)$$

Put $U(a,\ s) = q$ from eq. (5) in eq. (7), we get

$$U(a,\ s) = C_2\ e^{-a\sqrt{s/k}} = q$$

$$\Rightarrow \quad C_2 = q\ e^{a\sqrt{s/k}} \qquad (8)$$

Put C_2 from eq. (8) in eq. (7), we get

$$U(x, s) = q\, e^{a\sqrt{s/k}}\, e^{-x\sqrt{s/k}} = q\, e^{-(x-a)\sqrt{s/k}} \tag{9}$$

Taking Inverse Laplace transform of both sides of eq. (9), we get

$$u(x, t) = L^{-1}\{U(x, s)\} = L^{-1}\{q\, e^{-(x-a)\sqrt{s/k}}\}$$

$$= q\, L^{-1}\{e^{-(x-a)\sqrt{s/k}}\}$$

$$= \frac{q}{2\sqrt{\pi k t}}\, e^{-(x-a)^2/4kt} \qquad \left[\because\ L^{-1}\left\{e^{-b\sqrt{s/k}}\right\} = e^{-b^2/4kt}\right] \tag{10}$$

the required temperature.