

Lecture : 2

(INVERSE LAPLACE TRANSFORMS)

Course : B.Sc. (H) Physics
Semester : IV
Subject : Mathematical Physics III
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Topics to be covered:-

- **Inverse LT**
- **Inverse LT of Elementary Functions**
- **Properties of Inverse LT: Change of Scale Theorem, Shifting Theorem, Inverse LT of Derivatives and Integrals of Functions, Multiplication and Division by powers of s**
- **Convolution theorem**

Books to be referred:

1. Schaum's Outline of Theory and Problems of Laplace Transforms by Murray R. Spiegel.
2. Advanced Engineering Mathematics by Erwin Kreyzig
3. Mathematical Physics by H. K. Dass

2.1 Definition of Inverse Laplace Transformation:

If the Laplace Transform of $f(t)$ is $F(s)$, i.e. if $L\{f(t)\} = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ i.e.

$$\boxed{L^{-1}\{F(s)\} = f(t)}$$

where, L^{-1} is called the inverse Laplace transformation operator.

2.2 Inverse Laplace Transform of some elementary functions:

S. No.	$F(s)$	$L^{-1}\{F(s)\} = f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}}; n = 0, 1, 2, \dots$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-1}$	e^t
5.	$\frac{1}{s^2+1}$	$\sin t$
6.	$\frac{s}{s^2+1}$	$\cos t$
7.	$\frac{1}{s^2-1}$	$\sinh t$
8.	$\frac{s}{s^2-1}$	$\cosh t$

2.3 Change of Scale Property:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

Proof: By definition, we have $L\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt = F(s) \Rightarrow L^{-1}\{F(s)\} = f(t)$

$$\therefore L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^{\infty} f\left(\frac{t}{a}\right) e^{-st} dt \quad \text{Let } \frac{t}{a} = u \quad \therefore dt = a du$$

$$\Rightarrow L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^{\infty} f(u) e^{-s au} a du = a F(as)$$

Taking Inverse Laplace Transform on both sides, we get

$$\Rightarrow f\left(\frac{t}{a}\right) = L^{-1}[a F(as)]$$

$$\text{i. e., } L^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Similarly, we can prove that

$$L^{-1}[F(s/a)] = a f(ta)$$

Examples:

$$1. \quad \text{If } L^{-1}\left\{\frac{1}{s^2 + 64}\right\} = \frac{\sin 8t}{8}, \text{ find } L^{-1}\left\{\frac{1}{s^2 + 4}\right\} \text{ where } a > 0$$

$$\text{Sol. Given } L^{-1}\left\{\frac{1}{s^2 + 64}\right\} = \frac{\sin 8t}{8}$$

$$\therefore L^{-1}\left\{\frac{1}{(4s)^2 + 64}\right\} = \frac{1}{4} \frac{\sin (8t/4)}{8} \quad \left[\because L^{-1}\{F(as)\} = (1/a) f(t/a)\right]$$

$$\Rightarrow \frac{1}{16} L^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{16} \frac{\sin 2t}{2}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{\sin 2t}{2}$$

$$2. \quad \text{If } L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}, \text{ find } L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} \text{ where } a > 0$$

$$\text{Sol. Given } L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$$

$$\therefore L^{-1}\left\{\frac{e^{-a/s}}{(s/a)^{1/2}}\right\} = a \frac{\cos 2\sqrt{ta}}{\sqrt{\pi ta}} \quad \left[\because L^{-1}\{F(s/a)\} = a f(ta)\right]$$

$$\Rightarrow \sqrt{a} L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \sqrt{a} \frac{\cos 2\sqrt{ta}}{\sqrt{\pi t}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{ta}}{\sqrt{\pi t}}$$

2.4 Linearity Property:

If C_1 and C_2 are any constants, while $F_1(s)$ and $F_2(s)$ are Laplace Transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}\{C_1 F_1(s) + C_2 F_2(s)\} = C_1 L^{-1}\{F_1(s)\} + C_2 L^{-1}\{F_2(s)\} = C_1 f_1(t) + C_2 f_2(t)$$

Proof: By definition, we have $L\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt = F(s) \Rightarrow L^{-1}\{F(s)\} = f(t)$

$$\begin{aligned}
\text{Now, } L\{C_1 f_1(t) + C_2 f_2(t)\} &= \int_0^{\infty} [C_1 f_1(t) + C_2 f_2(t)] e^{-st} dt \\
&= C_1 \int_0^{\infty} f_1(t) e^{-st} dt + C_2 \int_0^{\infty} f_2(t) e^{-st} dt \\
&= C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\} \\
&= C_1 F_1(s) + C_2 F_2(s)
\end{aligned}$$

Taking Inverse Laplace Transform on both sides, we get

$$\begin{aligned}
\Rightarrow L^{-1}\{C_1 F_1(s) + C_2 F_2(s)\} &= C_1 f_1(t) + C_2 f_2(t) \\
\Rightarrow L^{-1}\{C_1 F_1(s) + C_2 F_2(s)\} &= C_1 L^{-1}\{F_1(s)\} + C_2 L^{-1}\{F_2(s)\}
\end{aligned}$$

Examples:

$$1. \quad \text{Find } L^{-1} \left\{ \frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4} \right\}$$

$$\begin{aligned}
\text{Sol. } L^{-1} \left\{ \frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4} \right\} \\
= L^{-1} \left\{ \frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+3^2} + \frac{18}{s^2+3^2} + \frac{24}{s^4} - \frac{30}{s^{7/2}} \right\} \\
= L^{-1} \left\{ \frac{5}{\Gamma(2)} \frac{\Gamma(2)}{s^2} + \frac{4}{\Gamma(3)} \frac{\Gamma(3)}{s^3} - \frac{2s}{s^2+3^2} + \frac{6 \times 3}{s^2+3^2} \right. \\
\left. + \frac{24}{\Gamma(4)} \frac{\Gamma(4)}{s^4} - \frac{30}{\Gamma(7/2)} \frac{\Gamma(7/2)}{s^{7/2}} \right\} \\
= \frac{5}{\Gamma(2)} L^{-1} \left\{ \frac{\Gamma(2)}{s^2} \right\} + \frac{4}{\Gamma(3)} L^{-1} \left\{ \frac{\Gamma(3)}{s^3} \right\} - 2 L^{-1} \left\{ \frac{s}{s^2+3^2} \right\} + 6 L^{-1} \left\{ \frac{3}{s^2+3^2} \right\} \\
+ \frac{24}{\Gamma(4)} L^{-1} \left\{ \frac{\Gamma(4)}{s^4} \right\} - \frac{30}{\Gamma(7/2)} L^{-1} \left\{ \frac{\Gamma(7/2)}{s^{7/2}} \right\} \\
= \frac{5}{1!} t + \frac{4}{2!} t^2 - 2 \cos 3t + 6 \sin 3t + \frac{24}{3!} t^3 - \frac{30}{(5/2)(3/2)(1/2)\sqrt{\pi}} t^{5/2} \\
\left[\because L^{-1} \left\{ \frac{\Gamma(n)}{s^n} \right\} = t^{n-1}, L^{-1} \left\{ \frac{s}{s^2+a^2} \right\} = \cos at, L^{-1} \left\{ \frac{a}{s^2+a^2} \right\} = \sin at \right] \\
= 5t + 2t^2 - 2 \cos 3t + 6 \sin 3t + 4t^3 - \frac{16}{\sqrt{\pi}} t^{5/2}
\end{aligned}$$

$$2. \quad \text{Find } L^{-1} \left\{ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right\}$$

$$\text{Sol. } L^{-1} \left\{ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right\}$$

$$\begin{aligned}
&= L^{-1} \left\{ \frac{6}{2(s-3/2)} - \frac{3}{9s^2-16} - \frac{4s}{9s^2-16} + \frac{8}{16s^2+9} - \frac{6s}{16s^2+9} \right\} \\
&= L^{-1} \left\{ \frac{6}{2(s-3/2)} - \frac{3}{9[s^2-(4/3)^2]} - \frac{4s}{9[s^2-(4/3)^2]} \right. \\
&\quad \left. + \frac{8}{16[s^2+(3/4)^2]} - \frac{6s}{16[s^2+(3/4)^2]} \right\} \\
&= L^{-1} \left\{ \frac{3}{s-3/2} - \frac{3}{9} \frac{3}{4} \frac{4/3}{s^2-(4/3)^2} - \frac{4}{9} \frac{s}{s^2-(4/3)^2} \right. \\
&\quad \left. + \frac{8}{16} \frac{4}{3} \frac{3/4}{s^2+(3/4)^2} - \frac{6}{16} \frac{s}{s^2+(3/4)^2} \right\} \\
&= 3 L^{-1} \left\{ \frac{1}{s-3/2} \right\} - \frac{1}{4} L^{-1} \left\{ \frac{4/3}{s^2-(4/3)^2} \right\} - \frac{4}{9} L^{-1} \left\{ \frac{s}{s^2-(4/3)^2} \right\} \\
&\quad + \frac{2}{3} L^{-1} \left\{ \frac{3/4}{s^2+(3/4)^2} \right\} - \frac{3}{8} L^{-1} \left\{ \frac{s}{s^2+(3/4)^2} \right\} \\
&= 3 e^{3t/2} - \frac{1}{4} \sinh \frac{4t}{3} - \frac{4}{9} \cosh \frac{4}{3} t + \frac{2}{3} \sin \frac{3t}{4} - \frac{3}{8} \cos \frac{3t}{4} \\
&\quad \left[\because L^{-1} \left\{ \frac{1}{s-a} \right\} = e^{at}, L^{-1} \left\{ \frac{s}{s^2-a^2} \right\} = \cosh at, L^{-1} \left\{ \frac{a}{s^2-a^2} \right\} = \sinh at, \right. \\
&\quad \left. L^{-1} \left\{ \frac{s}{s^2+a^2} \right\} = \cos at, L^{-1} \left\{ \frac{a}{s^2+a^2} \right\} = \sin at \right]
\end{aligned}$$

2.5 First Translation or Shifting Property:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s-b)\} = e^{bt} f(t)$

Proof: By definition, we have $F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt \Rightarrow L^{-1}\{F(s)\} = f(t)$

$$\therefore F(s-b) = \int_0^\infty f(t) e^{-(s-b)t} dt = \int_0^\infty \{e^{bt} f(t)\} e^{-st} dt = L\{e^{bt} f(t)\}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\{F(s-b)\} = e^{bt} f(t)$$

$$\text{i. e., } L^{-1}\{F(s-b)\} = e^{bt} L^{-1}\{F(s)\}$$

Examples:

$$1. \quad \text{Find } L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\}$$

$$\text{Sol. } L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\} = L^{-1} \left\{ \frac{6s-4}{(s-2)^2+16} \right\} = L^{-1} \left\{ \frac{6(s-2)+8}{(s-2)^2+4^2} \right\}$$

$$\begin{aligned}
&= 6 L^{-1} \left\{ \frac{s-2}{(s-2)^2 + 4^2} \right\} + \frac{8}{4} L^{-1} \left\{ \frac{4}{(s-2)^2 + 4^2} \right\} \\
&= 6 e^{2t} L^{-1} \left\{ \frac{s}{s^2 + 4^2} \right\} + 2 e^{2t} L^{-1} \left\{ \frac{4}{s^2 + 4^2} \right\} \quad [\because L^{-1} \{F(s-b)\} = e^{bt} L^{-1} \{F(s)\}] \\
&= 6 e^{2t} \cos 4t + 2 e^{2t} \sin 4t \quad \left[\because L^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at, L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at \right] \\
&= 2 e^{2t} (3 \cos 4t + \sin 4t)
\end{aligned}$$

2. Find $L^{-1} \left\{ \frac{1}{\sqrt{2s+3}} \right\}$

Sol.
$$\begin{aligned}
L^{-1} \left\{ \frac{1}{\sqrt{2s+3}} \right\} &= L^{-1} \left\{ \frac{1}{\sqrt{2} (s+3/2)^{1/2}} \right\} = \frac{1}{\sqrt{2}} L^{-1} \left\{ \frac{1}{\sqrt{s - (-3/2)}} \right\} \\
&= \frac{1}{\sqrt{2}} e^{-3t/2} L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} \quad [\because L^{-1} \{F(s-b)\} = e^{bt} L^{-1} \{F(s)\}] \\
&= \frac{1}{\sqrt{2}} e^{-3t/2} \frac{1}{\sqrt{\pi}} L^{-1} \left\{ \frac{\sqrt{\pi}}{\sqrt{s}} \right\} \\
&= \frac{1}{\sqrt{2}} e^{-3t/2} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \quad \left[\because L^{-1} \left\{ \sqrt{\frac{\pi}{s}} \right\} = \frac{1}{\sqrt{t}} \right] \\
&= \frac{1}{\sqrt{2\pi}} e^{-3t/2} t^{-1/2}
\end{aligned}$$

OR

We know that
$$L \left\{ \frac{1}{\sqrt{t}} \right\} = \int_0^\infty t^{-1/2} e^{-st} dt = \frac{1}{s^{-(1/2)+1}} \Gamma \left(\frac{-1}{2} + 1 \right) = \sqrt{\frac{\pi}{s}}$$

$$\left[\because \int_0^\infty t^{-n} e^{-st} dt = \frac{\Gamma(n+1)}{s^{n+1}} \text{ \& } \Gamma(1/2) = \sqrt{\pi} \right]$$

$$\therefore L \left\{ \frac{e^{-at}}{\sqrt{t}} \right\} = \sqrt{\frac{\pi}{s+a}} \quad \left[\because \text{if } L \{f(t)\} = F(s) \text{ then } L \{e^{at} f(t)\} = F(s-a) \right]$$

$$\Rightarrow L^{-1} \sqrt{\frac{\pi}{s+a}} = \frac{e^{-at}}{\sqrt{t}}$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\} = \frac{e^{-at}}{\sqrt{\pi t}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\} = \frac{1}{\sqrt{2}} \frac{e^{-at}}{\sqrt{\pi t}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} L^{-1} \left\{ \frac{1}{\sqrt{s+3/2}} \right\} = \frac{1}{\sqrt{2}} \frac{e^{-3t/2}}{\sqrt{\pi t}} \quad [\text{putting } a = 3/2]$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{\sqrt{2s+3}} \right\} = \frac{1}{\sqrt{2\pi}} e^{-3t/2} t^{-1/2}$$

3. Evaluate $L^{-1} \left\{ \frac{3s+7}{s^2-2s-3} \right\}$

Sol. $L^{-1} \left\{ \frac{3s+7}{s^2-2s-3} \right\} = L^{-1} \left\{ \frac{3s+7}{(s-1)^2-4} \right\} = L^{-1} \left\{ \frac{3(s-1)+10}{(s-1)^2-2^2} \right\}$

$$= 3 L^{-1} \left\{ \frac{(s-1)}{(s-1)^2-2^2} \right\} + \frac{10}{2} L^{-1} \left\{ \frac{2}{(s-1)^2-2^2} \right\}$$

$$= 3 e^t L^{-1} \left\{ \frac{s}{s^2-2^2} \right\} + 5 e^t L^{-1} \left\{ \frac{2}{s^2-2^2} \right\} \quad [\because L^{-1} \{F(s-b)\} = e^{bt} L^{-1} \{F(s)\}]$$

$$= 3 e^t \cosh 2t + 5 e^t \sinh 2t \quad \left[\because L^{-1} \left\{ \frac{s}{s^2-a^2} \right\} = \cosh at, L^{-1} \left\{ \frac{a}{s^2-a^2} \right\} = \sinh at \right]$$

2.6 Second Translation or Shifting Property:

If $L^{-1} \{F(s)\} = f(t)$ then $L^{-1} \{e^{-as} F(s)\} = g(t)$,

where, $g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt \quad \Rightarrow \quad L^{-1}\{F(s)\} = f(t)$$

$$\therefore e^{-as} F(s) = e^{-as} \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-s(t+a)} dt$$

$$= \int_0^{\infty} f(t+a-a) e^{-s(t+a)} dt \quad \text{Let } t+a=u \quad \therefore \quad dt=du$$

$$= \int_a^{\infty} f(u-a) e^{-su} du$$

$$= \int_a^{\infty} f(t-a) e^{-st} dt$$

$$= \int_0^a (0) e^{-st} dt + \int_a^{\infty} f(t-a) e^{-st} dt$$

$$= \int_0^{\infty} g(t) e^{-st} dt; \quad g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$$

$$= L\{g(t)\}$$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1} \{ e^{-as} F(s) \} = g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$$

Examples:

1. Find $L^{-1} \left\{ \frac{e^{-5s}}{(s-2)^4} \right\}$

Sol. Let us first find $L^{-1} \left\{ \frac{1}{(s-2)^4} \right\}$:

$$\begin{aligned} L^{-1} \left\{ \frac{1}{(s-2)^4} \right\} &= e^{2t} L^{-1} \left\{ \frac{1}{s^4} \right\} \quad [\because L^{-1} \{ F(s-b) \} = e^{bt} L^{-1} \{ F(s) \}] \\ &= e^{2t} \frac{1}{\Gamma(4)} L^{-1} \left\{ \frac{\Gamma(4)}{s^4} \right\} \\ &= e^{2t} \frac{1}{3!} t^3 \quad \left[\because L^{-1} \left\{ \frac{\Gamma(n)}{s^n} \right\} = t^{n-1} \right] \\ &= \frac{1}{6} e^{2t} t^3 \end{aligned}$$

By Using Second Translation Property, we get

$$L^{-1} \left\{ \frac{e^{-5s}}{(s-2)^4} \right\} = \begin{cases} \frac{1}{6} e^{2(t-5)} (t-5)^3 & ; \quad t > 5 \\ 0 & ; \quad t < 5 \end{cases}$$

2. Find $L^{-1} \left\{ \frac{(s+1) e^{-\pi s}}{s^2 + s + 1} \right\}$

Sol. Let us first find $L^{-1} \left\{ \frac{s+1}{s^2 + s + 1} \right\}$:

$$\begin{aligned} L^{-1} \left\{ \frac{s+1}{s^2 + s + 1} \right\} &= L^{-1} \left\{ \frac{s+1/2+1/2}{(s+1/2)^2 + 3/4} \right\} \\ &= L^{-1} \left\{ \frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2} \right\} + \frac{1}{2} \frac{2}{\sqrt{3}} L^{-1} \left\{ \frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2} \right\} \\ &= e^{-t/2} L^{-1} \left\{ \frac{s}{s^2 + (\sqrt{3}/2)^2} \right\} + \frac{1}{\sqrt{3}} e^{-t/2} L^{-1} \left\{ \frac{\sqrt{3}/2}{s^2 + (\sqrt{3}/2)^2} \right\} \\ &\quad [\because L^{-1} \{ F(s-b) \} = e^{bt} L^{-1} \{ F(s) \}] \end{aligned}$$

$$\begin{aligned}
&= e^{-t/2} \cos \frac{t\sqrt{3}}{2} + \frac{1}{\sqrt{3}} e^{-t/2} \sin \frac{t\sqrt{3}}{2} \\
&= e^{-t/2} \left[\cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \right]
\end{aligned}$$

By Using Second Translation Property, we get

$$L^{-1} \left\{ \frac{(s+1) e^{-\pi s}}{s^2 + s + 1} \right\} = \begin{cases} e^{-\frac{(t-\pi)}{2}} \left\{ \cos \frac{\sqrt{3}}{2} (t-\pi) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} (t-\pi) \right\}; & t > \pi \\ 0 & t < \pi \end{cases}$$

2.7 Inverse Laplace Transform of Derivatives:

If $L^{-1} \{ F(s) \} = f(t)$ then

$$\begin{aligned}
\text{(i)} \quad & L^{-1} \left\{ \frac{dF(s)}{ds} \right\} = -t f(t) \\
\text{(ii)} \quad & L^{-1} \left\{ \frac{d^2 F(s)}{ds^2} \right\} = (-1)^2 t^2 f(t)
\end{aligned}$$

Proof: (i) By definition, we have

$$\begin{aligned}
F(s) &= L \{ f(t) \} = \int_0^{\infty} f(t) e^{-st} dt \\
\therefore \quad \frac{dF(s)}{ds} &= \frac{d}{ds} \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} f(t) (-t) e^{-st} dt \\
&= - \int_0^{\infty} [t f(t)] e^{-st} dt \\
&= - L \{ t f(t) \}
\end{aligned} \tag{2.1}$$

Taking Inverse Laplace Transform on both sides, we get

$$\begin{aligned}
& L^{-1} \left\{ \frac{dF(s)}{ds} \right\} = -t f(t) \\
\text{i. e.,} \quad & L^{-1} \left\{ \frac{dF(s)}{ds} \right\} = -t L^{-1} \{ F(s) \} \\
\text{i. e.,} \quad & L^{-1} \{ F(s) \} = \frac{-1}{t} L^{-1} \left\{ \frac{dF(s)}{ds} \right\}
\end{aligned}$$

(ii) Differentiating both sides of eq. (2.1) w. r. t. s , we get

$$\begin{aligned}
\frac{d^2 F(s)}{ds^2} &= - \frac{d}{ds} \int_0^{\infty} [t f(t)] e^{-st} dt = - \int_0^{\infty} (-t) [t f(t)] e^{-st} dt \\
&= (-1)^2 \int_0^{\infty} [t^2 f(t)] e^{-st} dt \\
&= (-1)^2 L \{ t^2 f(t) \}
\end{aligned}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1} \left\{ \frac{d^2 F(s)}{ds^2} \right\} = (-1)^2 t^2 f(t)$$

Generalizing, $L^{-1} \left\{ \frac{d^n F(s)}{ds^n} \right\} = (-1)^n t^n f(t)$

Examples:

1. Find the inverse Laplace transform of $\cot^{-1} \left(\frac{s+3}{2} \right)$

Sol.
$$\begin{aligned} L^{-1} \left\{ \cot^{-1} \left(\frac{s+3}{2} \right) \right\} &= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \cot^{-1} \left(\frac{s+3}{2} \right) \right] & \left[\because L^{-1} \{F(s)\} = \frac{-1}{t} L^{-1} \left\{ \frac{dF(s)}{ds} \right\} \right] \\ &= \frac{-1}{t} L^{-1} \left\{ \frac{-(1/2)}{1 + (s+3)^2/2^2} \right\} \\ &= \frac{1}{2t} L^{-1} \left\{ \frac{2^2}{(s+3)^2 + 2^2} \right\} \\ &= \frac{1}{2t} 2 L^{-1} \left\{ \frac{2}{(s+3)^2 + 2^2} \right\} \\ &= \frac{1}{t} e^{-3t} L^{-1} \left\{ \frac{2}{s^2 + 2^2} \right\} & [\because L^{-1} \{F(s-b)\} = e^{bt} L^{-1} \{F(s)\}] \\ &= \frac{e^{-3t}}{t} \sin 2t & \left[\because L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at \right] \end{aligned}$$

2. Find $L^{-1} \left[\log \left(1 + \frac{1}{s^2} \right) \right]$

Sol.
$$\begin{aligned} L^{-1} \left[\log \left(1 + \frac{1}{s^2} \right) \right] &= L^{-1} \left[\log \left(\frac{s^2 + 1}{s^2} \right) \right] = L^{-1} [\log(s^2 + 1) - 2 \log s] \\ &= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \{ \log(s^2 + 1) - 2 \log s \} \right] & \left[\because L^{-1} \{F(s)\} = \frac{-1}{t} L^{-1} \left\{ \frac{dF(s)}{ds} \right\} \right] \\ &= \frac{-1}{t} L^{-1} \left[\frac{2s}{s^2 + 1} - \frac{2}{s} \right] \\ &= \frac{-2}{t} L^{-1} \left[\frac{s}{s^2 + 1^2} - \frac{1}{s} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{-2}{t} (\cos t - 1) \\
&= \frac{2}{t} (1 - \cos t)
\end{aligned}$$

$$\left[\because L^{-1} \left\{ \frac{1}{s} \right\} = 1, L^{-1} \left\{ \frac{s}{s^2 + 1} \right\} = \cos t \right]$$

2.8 Inverse Laplace Transform of Integrals:

If $L^{-1} \{ F(s) \} = f(t)$ then $L^{-1} \left\{ \int_s^\infty F(u) du \right\} = \frac{f(t)}{t}$

Proof: By definition, we have

$$\begin{aligned}
F(s) &= L \{ f(t) \} = \int_0^\infty f(t) e^{-st} dt \\
\therefore \int_s^\infty F(u) du &= \int_s^\infty \left(\int_0^\infty f(t) e^{-ut} dt \right) du \\
&= \int_0^\infty f(t) \left(\int_s^\infty e^{-ut} du \right) dt \\
&= \int_0^\infty f(t) \left(\frac{e^{-ut}}{-t} \right)_s^\infty dt \\
&= \int_0^\infty \frac{-f(t)}{t} [0 - e^{-st}] dt \\
&= \int_0^\infty \frac{f(t)}{t} e^{-st} dt \\
&= L \left\{ \frac{f(t)}{t} \right\}
\end{aligned}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1} \left\{ \int_s^\infty F(u) du \right\} = \frac{f(t)}{t}$$

Examples:

1. If $L^{-1} \left\{ \frac{1}{s(s+1)} \right\} = 1 - e^{-t}$, find $L^{-1} \left\{ \int_s^\infty \frac{du}{u(u+1)} \right\}$.

Sol. Given $L^{-1} \left\{ \frac{1}{s(s+1)} \right\} = 1 - e^{-t}$

$$\therefore L^{-1} \left\{ \int_s^\infty \frac{du}{u(u+1)} \right\} = \frac{1}{t} (1 - e^{-t}) \quad \left[\because \text{if } L^{-1} \{ F(s) \} = f(t), L^{-1} \left\{ \int_s^\infty F(u) du \right\} = \frac{f(t)}{t} \right]$$

2. If $L^{-1} \left\{ \frac{2a^2 s}{s^4 + 4a^4} \right\} = \sinh at \sin at$, find $L^{-1} \left\{ \int_s^\infty \frac{u du}{u^4 + 4a^4} \right\}$.

Sol. Given $L^{-1} \left\{ \frac{2a^2 s}{s^4 + 4a^4} \right\} = \sinh at \sin at$

$\therefore L^{-1} \left\{ \int_s^\infty \frac{2a^2 u du}{u^4 + 4a^4} \right\} = \frac{1}{t} \sinh at \sin at$

$$\left[\because \text{if } L^{-1}\{F(s)\} = f(t), L^{-1} \left\{ \int_s^\infty F(u) du \right\} = \frac{f(t)}{t} \right]$$

$\Rightarrow L^{-1} \left\{ \int_s^\infty \frac{u du}{u^4 + 4a^4} \right\} = \frac{1}{2a^2 t} \sinh at \sin at$

2.9 Multiplication by the powers of s :

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{s F(s) - f(0)\} = f'(t)$

If $f(0) = 0$, then $L^{-1}\{s F(s)\} = f'(t)$

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

$$\begin{aligned} \therefore L\{f'(t)\} &= \int_0^\infty f'(t) e^{-st} dt = \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty -s e^{-st} f(t) dt \\ &= 0 - f(0) + s L\{f(t)\} \\ &= s F(s) - f(0) \end{aligned}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\{s F(s) - f(0)\} = f'(t)$$

clearly, if $f(0) = 0$, then $L^{-1}\{s F(s)\} = f'(t)$

Examples:

1. Find $L^{-1} \left\{ \frac{s}{s^2 - 6s + 25} \right\}$.

Sol. Let us first find $L^{-1} \left\{ \frac{1}{s^2 - 6s + 25} \right\}$:

We have, $L^{-1} \left\{ \frac{1}{s^2 - 6s + 25} \right\} = L^{-1} \left\{ \frac{1}{(s-3)^2 + 4^2} \right\} = \frac{1}{4} L^{-1} \left\{ \frac{4}{(s-3)^2 + 4^2} \right\}$

$$= \frac{1}{4} e^{3t} L^{-1} \left\{ \frac{4}{s^2 + 4^2} \right\} \quad [\because L^{-1}\{F(s-b)\} = e^{bt} L^{-1}\{F(s)\}]$$

$$= \frac{1}{4} e^{3t} \sin 4t = f(t) \text{ and } f(0) = 0 \quad \left[\because L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at \right]$$

$\therefore L^{-1} \left\{ s \frac{1}{s^2 - 6s + 25} \right\} = f'(t) = \frac{d}{dt} \left(\frac{e^{3t}}{4} \sin 4t \right)$

$$\Rightarrow L^{-1} \left\{ \frac{s}{s^2 - 6s + 25} \right\} = \frac{1}{4} (e^{3t} 4 \cos 4t + 3 e^{3t} \sin 4t)$$

$$\Rightarrow L^{-1} \left\{ \frac{s}{s^2 - 6s + 25} \right\} = \frac{1}{4} e^{3t} (4 \cos 4t + 3 \sin 4t)$$

2. Find $L^{-1} \left\{ \frac{s^2}{(s^2 + 4)^2} \right\}$.

Sol. Let us first find $L^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\}$:

We have, $L^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{-1}{t} L^{-1} \left\{ \frac{d}{ds} \left(\frac{1}{s^2 + 4} \right) \right\} \quad \left[\because L^{-1} \{F(s)\} = \frac{-1}{t} L^{-1} \left\{ \frac{dF(s)}{ds} \right\} \right]$

$$\Rightarrow \frac{1}{2} L^{-1} \left\{ \frac{2}{s^2 + 2^2} \right\} = \frac{-1}{t} L^{-1} \left\{ \frac{-2s}{(s^2 + 4)^2} \right\}$$

$$\Rightarrow \frac{1}{2} \sin 2t = \frac{2}{t} L^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\} \quad \left[\because L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at \right]$$

$$\Rightarrow L^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\} = \frac{t}{4} \sin 2t = f(t) \text{ and } f(0) = 0$$

$$\therefore L^{-1} \left\{ s \frac{s}{(s^2 + 4)^2} \right\} = f'(t) = \frac{d}{dt} \left(\frac{t}{4} \sin 2t \right)$$

$$\Rightarrow L^{-1} \left\{ \frac{s^2}{(s^2 + 4)^2} \right\} = \frac{1}{4} (t 2 \cos 2t + \sin 2t)$$

$$\Rightarrow L^{-1} \left\{ \frac{s^2}{(s^2 + 4)^2} \right\} = \frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t$$

2.10 Division by s :

If $L^{-1} \{F(s)\} = f(t)$, then $L^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t f(u) du$

Proof: Let $g(t) = \int_0^t f(u) du$

then $g'(t) = f(t)$, with $g(0) = 0$

$$\therefore L \{g'(t)\} = s L \{g(t)\} - g(0)$$

$$\Rightarrow L \{f(t)\} = s L \{g(t)\} - 0$$

$$\Rightarrow F(s) = s L \{g(t)\}$$

$$\Rightarrow L \{g(t)\} = \frac{F(s)}{s}$$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1} \left\{ \frac{F(s)}{s} \right\} = g(t) = \int_0^t f(u) du$$

Examples:

1. Evaluate $L^{-1} \left\{ \frac{1}{s^2 (s^2 + a^2)} \right\}$.

Sol. $L^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a} L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \frac{1}{a} \sin at = f(t)$

$\therefore L^{-1} \left\{ \frac{1}{s (s^2 + a^2)} \right\} = \int_0^t f(u) du = \int_0^t \frac{1}{a} \sin au du$

$$\left[\because \text{if } L^{-1} \{F(s)\} = f(t), \text{ then } L^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t f(u) du \right]$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{s (s^2 + a^2)} \right\} = \left[\frac{-1}{a^2} \cos au \right]_0^t = \left[\frac{-1}{a^2} (\cos at - 1) \right]$$

$$= \frac{1}{a^2} (1 - \cos at) = g(t)$$

$\therefore L^{-1} \left\{ \frac{1}{s^2 (s^2 + a^2)} \right\} = \int_0^t g(u) du = \int_0^t \frac{1}{a^2} (1 - \cos au) du$

$$\left[\because \text{if } L^{-1} \{F(s)\} = g(t), \text{ then } L^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t g(u) du \right]$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{s^2 (s^2 + a^2)} \right\} = \frac{1}{a^2} \left[u - \frac{\sin au}{a} \right]_0^t = \frac{1}{a^2} \left[t - \frac{\sin at}{a} \right]$$

$$= \frac{1}{a^3} [at - \sin at]$$

2. Given $L^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} = \frac{t}{2} \sin t$, find $L^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\}$.

Sol. Given $L^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} = \frac{t}{2} \sin t = f(t)$

$\therefore L^{-1} \left\{ \frac{1}{s} \frac{s}{(s^2 + 1)^2} \right\} = \int_0^t f(u) du = \int_0^t \frac{u}{2} \sin u du$

$$\left[\because \text{if } L^{-1} \{F(s)\} = f(t), \text{ then } L^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t f(u) du \right]$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\} = \frac{1}{2} [u(-\cos u) - 1(-\sin u)]_0^t$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\} = \frac{1}{2} [(-1)(t \cos t - 0) + (\sin t - 0)]$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{(s^2 + 1)^2} \right\} = \frac{1}{2} [\sin t - t \cos t]$$

3. Find $L^{-1} \left[\frac{1}{s} \log \left(1 + \frac{1}{s^2} \right) \right]$

Sol. We have found that

$$L^{-1} \left[\log \left(1 + \frac{1}{s^2} \right) \right] = \frac{2}{t} (1 - \cos t) = f(t)$$

$$\therefore L^{-1} \left[\frac{1}{s} \log \left(1 + \frac{1}{s^2} \right) \right] = \int_0^t f(u) du = \int_0^t \frac{2(1 - \cos u)}{u} du$$

2.11 Convolution theorem:

If $f(t)$ and $g(t)$ are two functions and if $L^{-1} \{ F(s) \} = f(t)$ and $L^{-1} \{ G(s) \} = g(t)$,

$$\text{then } L^{-1} \{ F(s) G(s) \} = f(t) * g(t) = \int_0^t f(u) g(t-u) du = \int_0^t f(t-u) g(u) du$$

where, $f * g$ is called the convolution of f and g

Proof: By definition, we have

$$F(s) = L \{ f(t) \} = \int_0^\infty f(t) e^{-st} dt \quad \text{and} \quad G(s) = L \{ g(t) \} = \int_0^\infty g(t) e^{-st} dt$$

$$\begin{aligned} \therefore F(s) G(s) &= \left\{ \int_0^\infty f(u) e^{-su} du \right\} \left\{ \int_0^\infty g(v) e^{-sv} dv \right\} \\ &= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v) du dv && \text{Let } u + v = t \quad \therefore dv = dt \\ &= \int_0^\infty dt \int_0^t du e^{-st} f(u) g(t-u) \\ &= \int_0^\infty dt e^{-st} \left\{ \int_0^t du f(u) g(t-u) \right\} \\ &= L \left\{ \int_0^t f(u) g(t-u) du \right\} \\ &= L \{ f(t) * g(t) \} \end{aligned}$$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1} \{ F(s) G(s) \} = f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

Properties of Convolution:

1. Commutativity $f * g = g * f$

Proof: LHS = $f(t) * g(t) = \int_0^t f(u) g(t-u) du$

$$= \int_t^0 f(t-y) g(y) (-dy) \quad [\text{Putting } t-u = y \therefore du = -dy]$$

$$= \int_0^t f(t-y) g(y) dy$$

$$= \int_0^t f(t-u) g(u) du$$

$$= \int_0^t g(u) f(t-u) du = g(t) * f(t) = \text{RHS}$$

Hence, the convolution of f and g obeys the commutative law.

2. Associativity $f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$

Proof: Let $f(t) * [g(t) * h(t)] = f(t) * m(t)$, where $m(t) = g(t) * h(t)$

By Definition, $m(t) = g(t) * h(t) = \int_0^t g(u) h(t-u) du = \int_0^t h(u) g(t-u) du$

$$\therefore f(t) * m(t) = \int_0^t f(y) m(t-y) dy$$

$$\Rightarrow f(t) * [g(t) * h(t)] = \int_0^t f(y) dy \left[\int_0^{t-y} h(u) g(t-y-u) du \right]$$

$$= \int_0^t h(u) du \int_0^{t-u} f(y) g(t-y-u) dy$$

[changing the order of integration]

$$= h(t) * [f(t) * g(t)]$$

$$= [f(t) * g(t)] * h(t) \quad [\text{using commutativity property}]$$

3. Distributive with respect to addition $f * (g + h) = f * g + f * h$

Proof: $f * (g + h) = \int_0^t f(u) [g(t-u) + h(t-u)] du$

$$= \int_0^t f(u) g(t-u) du + \int_0^t f(u) h(t-u) du$$

$$= f * g + f * h$$

Examples:

1. Use Convolution Theorem to find $L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\}$.

Sol. Let $F(s) = \frac{1}{(s^2 + a^2)}, \quad G(s) = \frac{s}{(s^2 + a^2)}$

$\therefore f(t) = L^{-1} \{ F(s) \} = L^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a} \sin at$

and $g(t) = L^{-1} \{ G(s) \} = L^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at$

Now, using convolution theorem, we get

$$\begin{aligned} L^{-1} \{ F(s) G(s) \} &= \int_0^t f(u) g(t-u) du \\ \Rightarrow L^{-1} \left\{ \frac{1}{(s^2 + a^2)} \cdot \frac{s}{(s^2 + a^2)} \right\} &= \int_0^t \frac{1}{a} \sin au [\cos a(t-u)] du \\ \Rightarrow L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} &= \frac{1}{2a} \int_0^t 2 \sin au \cos(at-au) du \\ &= \frac{1}{2a} \int_0^t [\sin(au + at - au) + \sin(au - at + au)] du \\ &= \frac{1}{2a} \int_0^t [\sin(at) + \sin(2au - at)] du \\ &= \frac{1}{2a} \left[\sin(at) \int_0^t du + \int_0^t \sin(2au - at) du \right] \\ &= \frac{1}{2a} \left[t \sin(at) - \frac{\cos(2au - at)}{2a} \Big|_0^t \right] \\ &= \frac{t \sin at}{2a} - \frac{(\cos at - \cos at)}{4a^2} \\ &= \frac{t \sin at}{2a} \end{aligned}$$

2. Use Convolution Theorem to find $L^{-1} \left\{ \frac{1}{s^2 (s+1)^2} \right\}$.

Sol. Let $F(s) = \frac{1}{s^2}, \quad G(s) = \frac{1}{(s+1)^2}$

$\therefore f(t) = L^{-1} \{ F(s) \} = L^{-1} \left\{ \frac{1}{s^2} \right\} = t$

and $g(t) = L^{-1} \{ G(s) \} = L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} = t e^{-t}$

Now, using convolution theorem, we get

$$\begin{aligned}
L^{-1} \{ F(s) G(s) \} &= \int_0^t f(u) g(t-u) du \\
\Rightarrow L^{-1} \left\{ \frac{1}{s^2} \cdot \frac{1}{(s+1)^2} \right\} &= \int_0^t u e^{-u} (t-u) du \\
&= t \int_0^t u e^{-u} du - \int_0^t u^2 e^{-u} du \\
&= t [-u e^{-u} - e^{-u}]_0^t - [-u^2 e^{-u} - 2u e^{-u} - 2e^{-u}]_0^t \\
&= t [-t e^{-t} - (e^{-t} - 1)] - [-t^2 e^{-t} - 2t e^{-t} - 2(e^{-t} - 1)] \\
&= t e^{-t} + 2e^{-t} + t - 2
\end{aligned}$$

Partial Fractions:

Sometimes the partial fraction method is very useful in finding the Inverse Laplace Transform.

Examples:

1. Evaluate $L^{-1} \left\{ \frac{s}{s^4 + s^2 + 1} \right\}$.

Sol. Now, $\frac{s}{s^4 + s^2 + 1} = \frac{s}{(s^2 + 1)^2 - s^2} = \frac{s}{(s^2 + 1 + s)(s^2 + 1 - s)}$
 $= \frac{1}{2} \left[\frac{1}{s^2 - s + 1} - \frac{1}{s^2 + s + 1} \right]$ [Resolving into partial fractions]

$$\begin{aligned}
\therefore L^{-1} \left\{ \frac{s}{s^4 + s^2 + 1} \right\} &= \frac{1}{2} L^{-1} \left[\frac{1}{s^2 - s + 1} - \frac{1}{s^2 + s + 1} \right] \\
&= \frac{1}{2} L^{-1} \left[\frac{1}{(s - 1/2)^2 + 3/4} - \frac{1}{(s + 1/2)^2 + 3/4} \right] \\
&= \frac{1}{2} e^{t/2} L^{-1} \left\{ \frac{1}{s^2 + (\sqrt{3}/2)^2} \right\} - \frac{1}{2} e^{-t/2} L^{-1} \left\{ \frac{1}{s^2 + (\sqrt{3}/2)^2} \right\} \\
&= \frac{1}{2} e^{t/2} \frac{2}{\sqrt{3}} L^{-1} \left\{ \frac{\sqrt{3}/2}{s^2 + (\sqrt{3}/2)^2} \right\} - \frac{1}{2} e^{-t/2} \frac{2}{\sqrt{3}} L^{-1} \left\{ \frac{\sqrt{3}/2}{s^2 + (\sqrt{3}/2)^2} \right\} \\
&= e^{t/2} \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) - e^{-t/2} \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) \\
&= \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) (e^{t/2} - e^{-t/2}) \\
&= \frac{2}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) \sinh \left(\frac{t}{2} \right)
\end{aligned}$$

2. Find $L^{-1} \left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\}$

Sol.
$$\begin{aligned} L^{-1} \left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\} &= L^{-1} \left\{ \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \right\} = L^{-1} \left\{ \frac{A}{s-1} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1} \right\} \\ &= A L^{-1} \left\{ \frac{1}{s-1} \right\} + B L^{-1} \left\{ \frac{s}{s^2+1} \right\} + C L^{-1} \left\{ \frac{1}{s^2+1} \right\} \\ &= A e^t + B \cos t + C \sin t \end{aligned} \quad (2.2)$$

Now to find the constants A , B and C :

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow 3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

Put $s = 1$, we get $4 = 2A \Rightarrow A = 2$

Comparing the coefficients of s^2 on both sides, we get

$$0 = A + B \Rightarrow B = -A = -2$$

Put $s = 0$, we get $1 = A - C \Rightarrow C = A - 1 = 2 - 1 = 1$

Put $A = 2$, $B = -2$ and $C = 1$ in eq. (2.2), we get

$$\therefore L^{-1} \left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\} = 2e^t - 2\cos t + \sin t$$

Note: $h(t) = \int_0^t g(u) du \Leftrightarrow \frac{\partial}{\partial t} h(t) = g(t), \text{ provided } h(0) = 0.$

Eg. $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \Leftrightarrow \frac{\partial}{\partial t} \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$

where $\operatorname{erf}(t)$ is known as error function.