Chap 10 Integration.

No.

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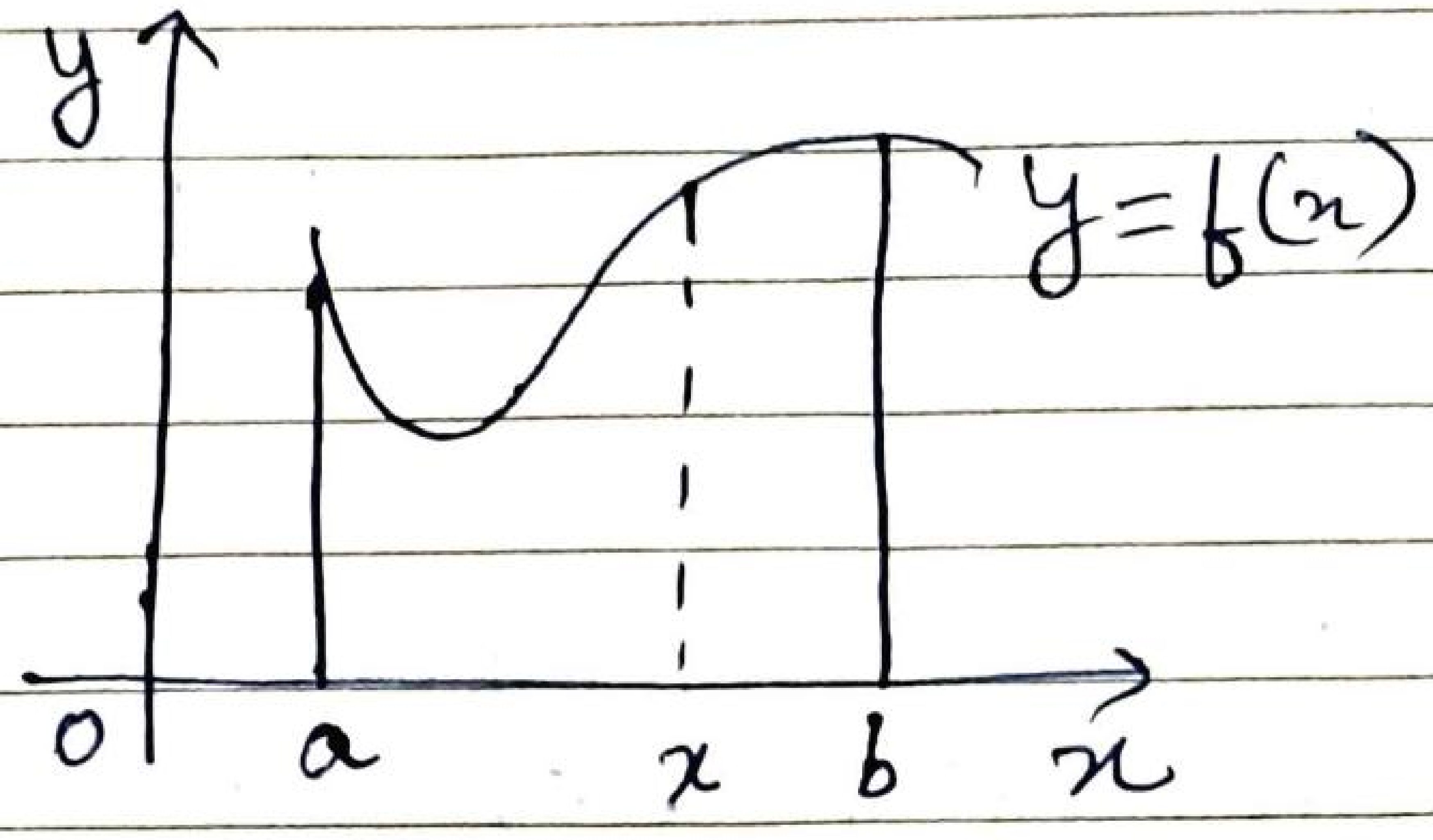
Suppose $y = f(x)$ where $f(x)$ is continuous and positive.



$A(a) = 0$
$A(b) = A$

$A(x)$ = area under the curve $f(x)$ over the interval $[a, x]$

$A'(x) = f(x) \quad \forall x \in (a, b)$
Derivative of the area function is the curve's height function

Antiderivative of f: A function F with the property that, $F'(x) = f(x) \; \forall x$ in some open interval, is often called antiderivative of f. Antiderivative are infinitely many:

$$\frac{d}{dx}\left(F(x) + C\right) = F'(x) = f(x)$$

real constant

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No.

Date

Area below $f(x)$ and above the x axis from $x = a$ to $x = b$.

(1) Find the antiderivative of $f(x)$

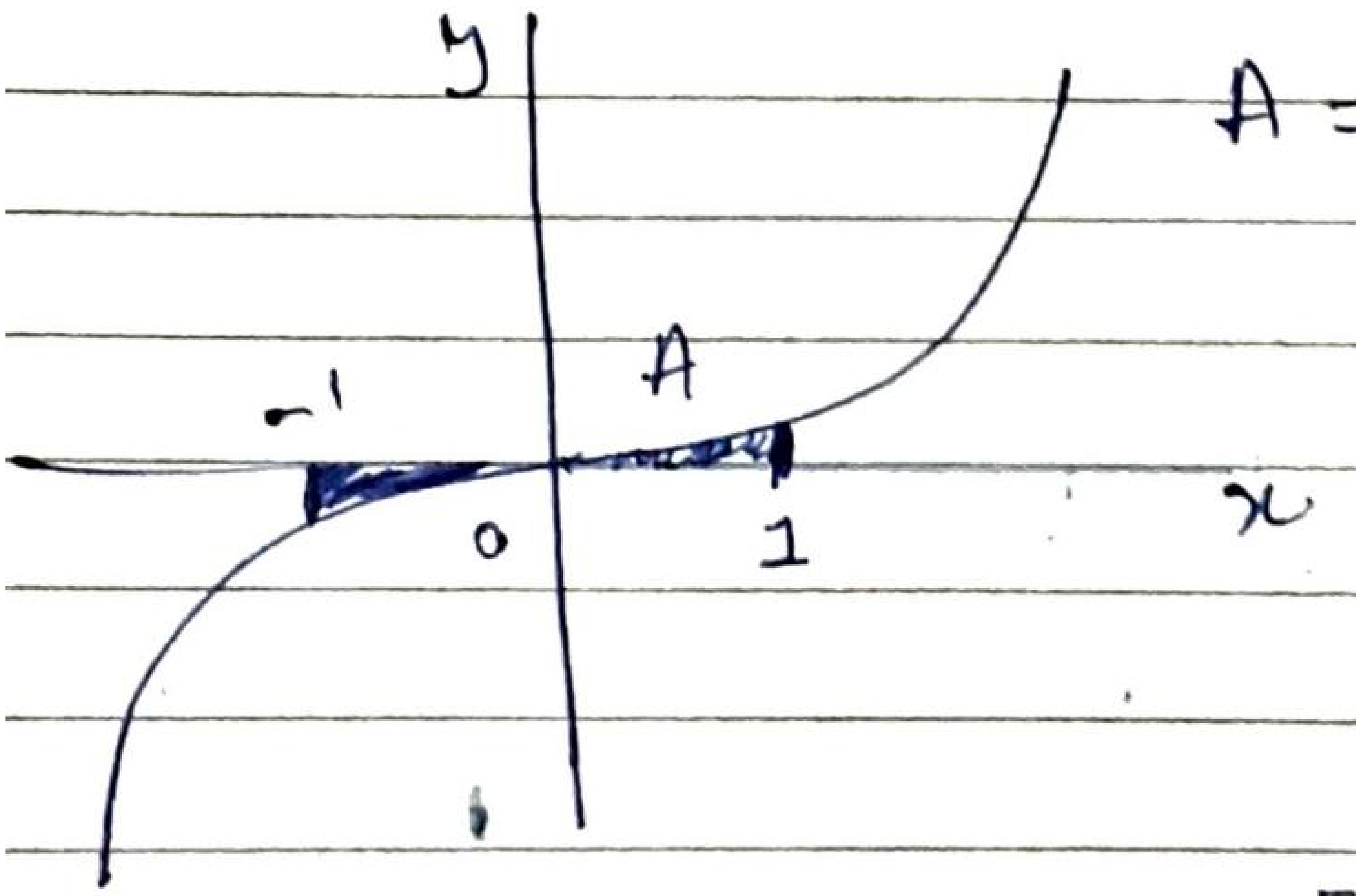
(2) The required area is:

$F(b) - F(a)$.

If $f(x)$ is continuous in $[a, b]$ with $f(x) \leq 0 \; \forall \; x \in [a, b]$, then area

$$= -\left[F(b) - F(a)\right].$$

$f(x) = x^3$ $[0, 1]$



$$A = \int_0^1 x^3 \, dx$$

$$= \left.\frac{x^4}{4}\right]_0^1$$

$$= 1/4$$

= area in $[-1, 0]$

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Antiderivative | Indefinite Integral of a function $f(x)$

$$\int f(x)\,dx = F(x) + C$$

integrand

constant of integration

where $F'(x) = f(x)$.

integral sign

variable of integration

RULES

① $\int x^a\,dx = \frac{1}{a+1} x^{a+1} + C \quad (a \neq -1)$

② $\int \frac{1}{x}\,dx = \ln|x| + C$

③ $\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C \quad (a \neq 0)$

④ $\int a^x\,dx = \frac{1}{\ln a} a^x + C$

$(a > 0 \text{ and } a \neq 1)$

⑤ $\int a f(x)\,dx = a \int f(x)\,dx$

(a is a real constant)

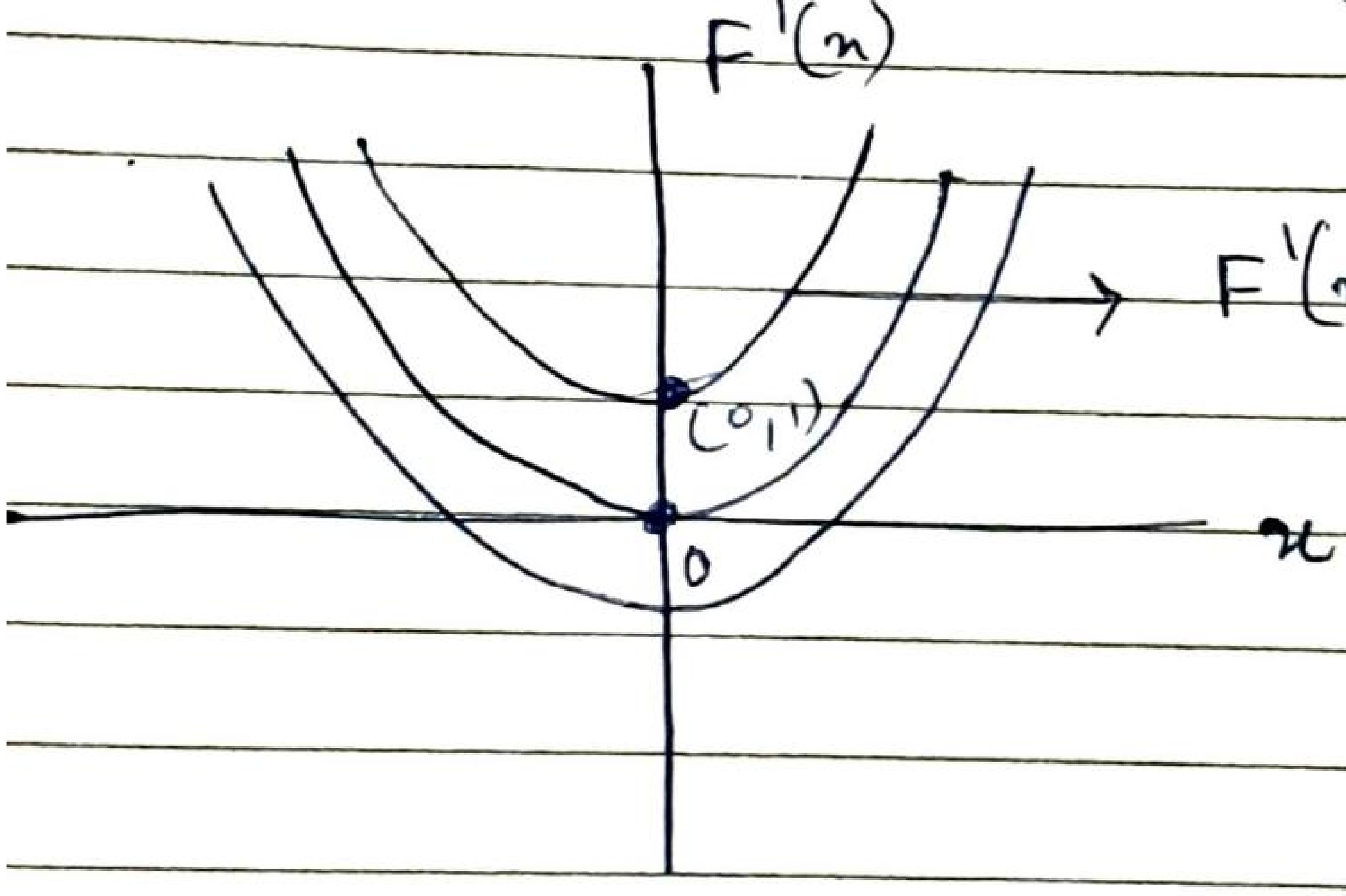
(6) $\int \left(f(x) + g(x)\right) dx = \int f(x)\,dx + \int g(x)\,dx$

<u>Initial value problem</u>

Suppose $f(x) = 2x$.
Find the function whose graph passes through $(0, 1)$

$F'(x) = 2\,\frac{x^2}{2} + C$

$F'(x) = x^2 + C \quad ; \left(F'(x) - C\right) = (x - 0)^2 \to (1)$



$\to F'(x) = x^2 + 1$

$(1,1)$ passes through (1)

$$1 - C = (0 - 0)^2$$
$$C = 1$$

$$F'(x) - 1 = x^2$$
$$F'(x) = x^2 + 1$$

Definite Integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ is called definite integral.

$$= F(x) \Big|_a^b$$

where $F'(x) = f(x) \quad \forall x \in (a,b)$

PROPERTIES :

① $$\int_a^b f(x)\,dx = - \int_b^a f(x)\,dx$$

② $$\int_a^a f(x)\,dx = 0$$

③ $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$ (α is an arbitrary no.)

④ $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Theorem : If f is continuous function in $[a,b]$, then there exist a continuous function $F(x)$ in $[a,b]$ such that $F'(x) = f(x) \quad \forall x \in (a,b)$

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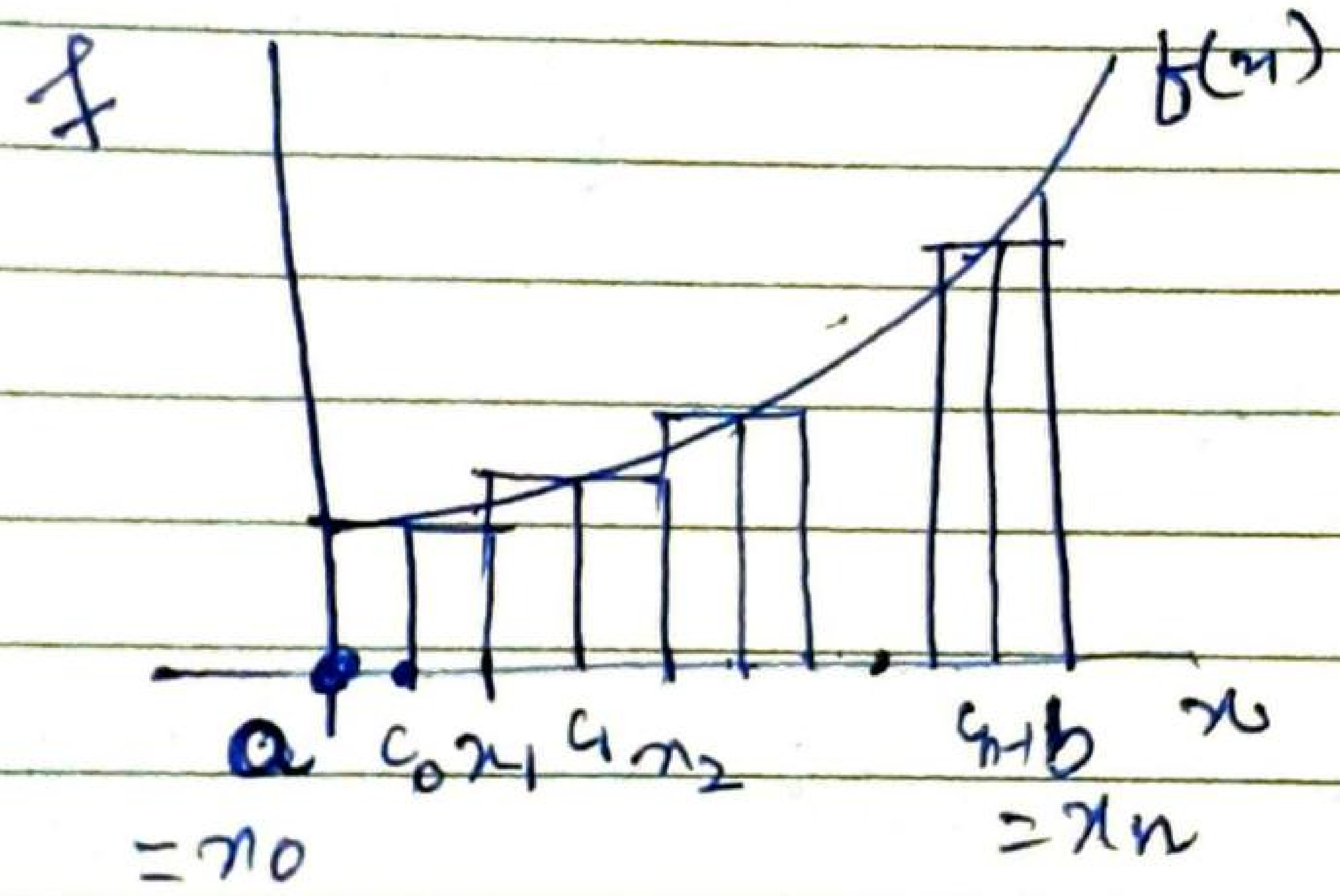
No.

Date

Riemann Integral.

Let f be a bounded function in $[a, b]$ and n be a natural number. We subdivide $[a, b]$ into n parts by choosing $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

Put $\Delta x_i = x_{i+1} - x_i$, $i = 0, \dots, n-1$ and choose an arbitrary number c_i in each interval $[x_i, x_{i+1}]$



$$\text{Riemann sum} = f(c_0)\Delta x_0 + f(c_1)\Delta x_1 + \dots + f(c_{n-1})\Delta x_{n-1}$$

f is called Riemann integrable in $[a, b]$ and we put:

No.

Date

$$\int_a^b f(x)\,dx = \lim \sum_{i=0}^{n-1} f(c_i)\,\Delta x_i$$

Approximation of area gets better when n increases.

Sec 10.4 Economic application of integration.

INCOME DISTRIBUTION

F(r) = proportion of individuals who recieve no more than r \$

r = income

n = no. of individuals in the population

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r is given to $\in (r_0, r_1)$
Let F be a function with a continuous derivative f, i.e.
$$f(r) = F'(r) \quad \forall\, r \in (r_0, r_1)$$

i.e. $$f(r) = \lim_{\Delta r \to 0} \frac{F(r+\Delta r) - F(r)}{\Delta r}$$

f = income density function
F = cumulative distribution function

If $r_0 \le a \le b \le r_1$

$\int_a^b f(r)\,dr$ = proportion of individuals with income in $[a, b]$

$n \int_a^b f(r)\,dr$ = number of individuals with incomes in $[a, b]$

Let $M(r)$ = Total income of those who earn no more than r \$.

$$M'(r) = \lim_{\Delta r \to 0} \frac{M(r+\Delta r) - M(r)}{\Delta r}$$

$f(x)\,\Delta x \rightarrow$ approx equal to the prop^n of individuals who earn b/w x & $x+\Delta x$

$M'(x) \approx f(x)\,\Delta x \cdot n \cdot x$

$\cancel{= \lim_{\Delta x \to 0} n\,f(x+\Delta x)(x+\Delta x)}$

$= nx\,f(x)$

$M'(x) = nx\,f(x)$

$$\int_a^b M'(x)\,dx = n\int_a^b x\,f(x)\,dx$$

$$= M(b) - M(a)$$

Total income of individuals who have incomes in the interval [a, b] $= n\int_a^b x\,f(x)\,dx$

Mean income of individuals with incomes in the interval [a, b] $= m = \dfrac{n\int_a^b x\,f(x)\,dx}{n\int_a^b f(x)\,dx}$

$= \dfrac{\text{Total income}}{\text{Number of individual belonging to a within income interval } [a,b]}$

No.

Date

$D(p, r)$ = continuous function that denotes the number of commodity units demanded by an individual with income r and price p.

$T(r)$ = Total Demand for the commodity by all individuals whose incomes are $\leq r$, and price is fixed at 'p'.

Let income $\in [a, b]$

$$\int T'(r)\,dr \approx \underbrace{n f(r) \Delta r}_{\substack{\text{no. of inds} \\ \text{who earn} \\ \text{b/w } r \,\&\, r+\Delta r}} \cdot D(p, r)$$

$$x(p) = \int_a^b T'(r)\,dr = \int_a^b n D(p, r) f(r)\,dr$$

= Total demand of the commodity by all individuals whose incomes lie b/w 'a' and 'b', for a given price p.

PRESENT DISCOUNTED VALUE

Time = t

roi = r

$P(t)$ = Present discounted value of all payments made over the time interval $[0, t]$

$f(t)$ = rate at which income is received per year at time t

$$P'(t) = f(t)\, e^{-rt}$$

$$\int_0^T P'(t)\,dt = \int_0^T f(t)\, e^{-rt}\,dt$$

$$P(T) - P(0) = \int_0^T f(t) e^{-rt}\,dt$$

Present discounted value (at time 0) of a continuous income stream at the rate of $f(t)$ \$/year over the time interval $[0, T]$ with continuously compounded interest at rate r, is given by:

$$PDV = P(T) = \int_0^T f(t)\, e^{-rt}\,dt$$

Future value of this amount after T years =

$$\left(\int_0^T f(t)\, e^{-rt}\, dt\right) e^{rT}$$

$$= \int_0^T f(t)\, e^{-r(t-T)}\, dt.$$

$$= \int_0^T f(t)\, e^{r(T-t)}\, dt$$

Future discounted value (at time T) of a continuous income stream at the rate of $f(t)$ \$/year over the time interval $[0, T]$ with continuously compounded interest at rate r is given by

$$FDV = \int_0^T f(t)\, e^{r(T-t)}\, dt$$