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Sol We will solve it using Cauchy Conv. Criterion.

here x_1 & x_2 are unknown. & $x_2 > x_1$

We will show that the given seq. is a Cauchy seq. $\therefore$ it is cgt.

$\therefore$ T.S. $\rightarrow$ for $\epsilon > 0$, $\exists$ $H \in \mathbb{N}$: $|x_n - x_m| < \epsilon \quad \forall\ n, m \geq H$ (let $m > n$)

Now $|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \dots + x_{m-1} - x_m|$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \dots + |x_{m-1} - x_m| \quad \text{— (1)}$$

Since $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$

$\therefore |x_n - x_{n+1}| = \frac{1}{2^{n-1}} |x_2 - x_1|$ — (*) (See Pf at the end) using induction

$\therefore$ (1) $\Rightarrow$

$$|x_n - x_m| \leq \left[\frac{1}{2^{n-1}} + \frac{1}{2^n} + \dots + \frac{1}{2^{m-2}}\right] |x_2 - x_1|$$

$$= \frac{1}{2^{n-1}} \left[1 + \frac{1}{2} + \dots + \frac{1}{2^{m-n-1}}\right] |x_2 - x_1|$$

$$= \frac{1}{2^{n-1}} \left[\frac{1\left(1 - \left(\frac{1}{2}\right)^{m-n}\right)}{1 - \frac{1}{2}}\right] |x_2 - x_1|$$

$$< \frac{2}{2^{n-1}} |x_2 - x_1| \longrightarrow 0 \quad \text{as } n \to \infty$$

Since $m > n \Rightarrow m \to \infty$ $\quad \therefore |x_n - x_m| \to 0$ as $n, m \to \infty$

$\Rightarrow |x_n - x_m| < \epsilon \quad \forall\ n, m \geq H$

$\Rightarrow (x_n)$ is Cauchy seq. $\therefore$ Cgt.
Let it converge to l

To find l :
we have

$$\left.\begin{aligned} x_3 &= \tfrac{1}{2}(x_1+x_2) \\ x_4 &= \tfrac{1}{2}(x_2+x_3) \\ &\;\; - - - - \\ x_k &= \tfrac{1}{2}(x_{k-2}+x_{k-1}) \end{aligned}\right\} \text{add}$$

$$\Rightarrow x_3+x_4+\cdots+x_k = \tfrac{1}{2}x_1 + [x_2+x_3+\cdots+x_{k-2}] + \tfrac{1}{2}x_{k-1}$$

$$\Rightarrow \tfrac{1}{2}x_{k-1} + x_k = \tfrac{1}{2}x_1 + x_2$$

take the limit as $k \to \infty$

$$\Rightarrow \frac{l}{2} + l = \frac{1}{2}x_1 + x_2 \Rightarrow \frac{3}{2}l = \frac{1}{2}x_1 + x_2$$

$$\Rightarrow l = \frac{x_1+2x_2}{3}$$

Pf
(*) for $n=1$ $|x_1-x_2| = |x_2-x_1| = \frac{1}{2^0}|x_2-x_1|$

for $n=2$ $|x_2-x_3| = |x_2 - \frac{1}{2}(x_1+x_2)| = |\frac{1}{2}x_2 - \frac{1}{2}x_1|$

$$= \frac{1}{2}|x_2-x_1|$$

for $n=3$ $|x_3-x_4| = |x_3 - \frac{1}{2}(x_2+x_3)|$

$$= \frac{1}{2}|x_3-x_2|$$

$$= \frac{1}{2^2}|x_2-x_1| \quad \text{etc.}$$

ie. $|x_n - x_{n+1}| = \frac{1}{2^{n-1}}|x_2-x_1|$

Infinite Series

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$$

(here sign 'Σ' doesn't indicate the sum) $\because$ $u_1 + u_2 + \cdots$ upto infinity may not be summable)

the seq. $(u_1, u_2, u_3, \ldots)$ is called the seq. of the terms of the series.

Consider a seq. (S_n) where

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$- - - -$

$S_n = u_1 + u_2 + \cdots + u_n$, then the seq. (S_n) is

Called the seq. of the partial sums of the series (SOPS)

→ Convergence or divergence of the series Σu_n

A series Σu_n is said to be cgt or div. according as the seq. of partial sums (S_n) of the series is cgt or div.

→ In case (S_n) is cgt, its limit is the sum of the series Σu_n

In that case Σ is the sum of the series

Exp. (1) $\sum_{n=0}^{\infty} 2^n = 1+2+2^2+2^3+\cdots$

here $S_n = 1+2+2^2+\cdots+2^{n-1} = \frac{1(2^n-1)}{2-1} = 2^n-1$
$= \cancel{1(1-2^n)}$

$\to \infty$ as $n \to \infty$

Since $(S_n) \to \infty$ as $n \to \infty$

i.e. SOPS is div. $\therefore$ the given series is div.

i.e. it is not summable.

(2) $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$

here $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$

$\to 2$ as $n \to \infty$

$\therefore (S_n) \to 2$

Since SOPS converges to 2 $\therefore \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

Exp (3) $\sum u_n$

$= 2-2+2-2+2-\cdots$

here $S_1 = 2$

$S_2 = 2-2 = 0$ i.e. $S_{2n} = 0$

$S_3 = 2-2+2 = 2$ $S_{2n+1} = 2$

$S_4 = \underbrace{2-2}+\underbrace{2-2} = 0$ $\therefore S_n = (2, 0, 2, 0, \ldots)$

— — — — —

is a div. seq

$\therefore \sum u_n$ is a div. series