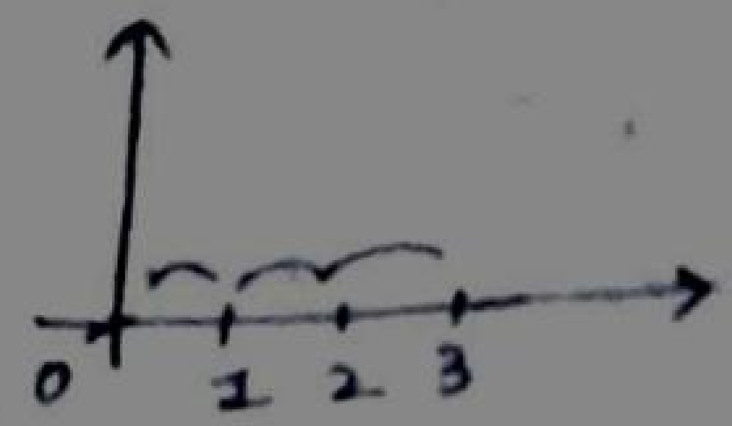
Section 20.1

DIFFERENCE EQUATIONS (Chap 20).

When time is taken to be a <u>discrete variable</u> then the equations that relate the value of such variables (say income, consumption, savings, Borrowings etc) at different points of time, are called <u>Difference Equations</u> or <u>Recurrence relations</u>.

$t = 0$ (present)



(when time is continuous and we consider equations involving variables and their Derivatives
⇓
DIFFERENTIAL EQ. Sem II)

Let $f(t, x)$ be a function defined for all positive integers t and all real numbers x. A <u>general DIFF</u> equation of the first order is

$$x_t = f(t, x_{t-1}) \qquad (t = 1, 2, 3, \ldots)$$

$$\Delta x_t = x_t - x_{t-1}$$
$$= f(t, x_{t-1}) - x_{t-1}$$

Suppose x_0 is given (value of variable in the current pd is given)

$$x_1 = f(1, x_0)$$
$$x_2 = f(2, x_1) = f(2, f(1, x_0))$$
$$x_3 = f(3, x_2) = f(3, f(2, x_1))$$
$$= f(3, f(2, f(1, x_0)))$$

$$x_4 =$$

$$x_5 =$$

⋮

So for a given value of x_0, we can compute x_t for any value of t.

Theorem Existence and uniqueness Theorem

Consider the difference equation $x_t = f(t, x_{t-1})$, where f is defined for all $t = 1, 2, \ldots$ values of the variables. If x_0 is an arbitrary fixed number, then there exist a uniquely determined function x_t that is a solution of the equation and has the given value for $t = 0$.

DEFECTS OF INSERTION METHOD

1. Time consuming to get an answer for say x_{100}
2. compilation errors can occur
3. we might be interested in the behaviour of the solution when t becomes very large or how changes in the parameters entering the difference equation affects the solution. etc.

$\therefore$

r_i – roi in pd i (given)

Say A_0 present amount

$A_1 = A_0(1+r_1)$

$A_2 = A_1(1+r_2) = A_0(1+r_1)(1+r_2)$

$A_3 = A_2(1+r_3) = A_0(1+r_1)(1+r_2)(1+r_3)$

$\vdots$

$A_t = A_{t-1}(1+r_t) = A_0 \prod_{i=1}^{t}(1+r_i)$

eg ① If $x_0 = 15$ & $x_t = x_{t-1} + 2$, Find $x_t = x_t(t)$.

$x_1 = 15+2 = x_0 + 2(1) = 17$
$x_2 = x_1+2 = x_0 + 2(2) = 19$
$x_3 = x_2+2 = x_0 + 2(3) = 21$

$x_t = x_0 + 2(t)$
$x_t = 15 + 2t \quad \forall t$

Linear fn of t

② If $x_t = 0.9\, x_{t-1}$ & x_0 given, find $x_t = x_t(t)$

$x_1 = 0.9 x_0$
$x_2 = 0.9 x_1 = (0.9)^2 x_0$
$x_3 = 0.9 x_2 = (0.9)^3 x_0$

Exponential fn of t

$x_t = (0.9)^t x_0 \quad$ = a function of t

Q3 The number of rabbits on a farm increases by 8% @ in addition to the removal of 4 rabbits per year for adoption. The farm starts out with 35 rabbits. Find population as a function of time

$y_0 = 35$; Popn after n years $= y_n$
$y_n = 1.08\, y_{n-1} - 4 \quad ; \; y_0 = 35$

$y_1 = 1.08(35) - 4$
$y_2 = 1.08\,[1.08(35) - 4] - 4 = (1.08)^2 35 - (1.08+1)4$
$= (1.08)^2 35 - (1.08)4 - 4$
$y_3 = 1.08\,[(1.08)^2(35) - (1.08+1)4] - 4$
$= (1.08)^3(35) - (1.08)^2 4 - (1.08)4 - 4$

$y_n = (1.08)^n (y_0) - 4\sum_{k=1}^{n} (1.08)^{k-1}$

Q4 Suppose an amount of Rs 1000 (A_0) is invested in a bank. The ri is 5% @. What is the amount after t years? $A_t = A_{t-1}(1+.05)$

$A_1 = 1000\,(1+5\%) = 1000\,(1.05)$

$A_2 = 1000\,(1+5\%)\,(1+5\%) = 1000\,(1+5\%)^2$

$A_3 = 1000\,(1.05)^3$

$$\boxed{A_t = 1000\,(1.05)^t} = A_0\,(1.05)^t$$

In Q1, Q2, Q3, Q4. we were given a difference equation and value of the dependent variable in the time period '0' and we could see that a unique function exist for the dependent variable which is a function of time. This is indeed the Solution of difference equation.

FIRST ORDER EQUATIONS WITH A CONSTANT COEFFICIENT

Linear Difference Equation : $x_t = a\,x_{t-1} + b_t$ $\quad (t = 1, 2, 3, \dots)$ — (1)

If x_0 is given, then

$$x_1 = a x_0 + b_1$$

$$x_2 = a x_1 + b_2 = a(a x_0 + b_1) + b_2 = a^2 x_0 + a b_1 + b_2$$

$$x_3 = a x_2 + b_3 = a(a^2 x_0 + a b_1 + b_2) + b_3$$
$$= a^3 x_0 + a^2 b_1 + a b_2 + b_3$$

$$x_4 = a x_3 + b_4 = a(a^3 x_0 + a^2 b_1 + a b_2 + b_3) + b_4$$
$$= a^4 x_0 + a^3 b_1 + a^2 b_2 + a b_3 + b_4$$

$$x_5 = a x_4 + b_5 = a^5 x_0 + a^4 b_1 + a^3 b_2 + a^2 b_3 + a b_4 + b_5$$

Pattern is clear for the solution:-

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k \quad (t = 1, 2, 3, \dots) \quad \text{— (2)}$$

(2) is a solution of (1) $\boxed{\text{PROOF}}$

Put (2) in RHS of (1)

$$x_{t-1} = a^{t-1} x_0 + \sum_{k=1}^{t-1} a^{t-1-k} b_k \quad \text{— From (1)}$$

or $$a x_{t-1} + b_t = a\left(a^{t-1} x_0 + \sum_{k=1}^{t-1} a^{t-1-k} b_k\right) + b_t$$

$$= a^t x_0 + \sum_{k=1}^{t-1} a^{t-k} \cdot b_k + b_t = a^t x_0 + \sum_{k=1}^{t-1} a^{t-k} \cdot b_k + a^{t-t} b_t$$

$$= a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k = x_t$$

Since the above matches the expression for x_t in (2); our sol^n is indeed the correct sol^n.

III

(Special case $b_k = b \quad \forall \; k = 1, 2, \ldots$

$$\sum_{k=1}^{t} a^{t-k} b_k = b \sum_{k=1}^{t} a^{t-k} = b\left(a^{t-1} + a^{t-2} + \ldots + a^2 + a + 1\right)$$

$$= b\left(\frac{1-a^t}{1-a}\right) \quad a \neq 1$$

so $x_t = a x_{t-1} + b \iff x_t = a^t (x_0) + b\left(\frac{1-a^t}{1-a}\right)$
& $a \neq 1$.

$$= a^t\left(x_0 - \frac{b}{1-a}\right) + \left(\frac{b}{1-a}\right)$$

ALSO;

$x_t = a x_{t-1} + b \iff x_t = x_0 + tb$
& $a = 1$

PROOF

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k$$

$$= a^t x_0 + b \sum_{k=1}^{t} a^{t-k} \quad \text{because } b_k = b \;\forall\, t$$

$$= x_0 + b\left[a^{t-1} + a^{t-2} + \ldots + a + 1\right]$$

OR

$$x_t = x_0 + bt$$

Find the sol^n to the following Diff Eq^n with given value of x_0.

(b) $3x_t = x_{t-1} + 2$; $x_0 = 2$

$x_t = \frac{1}{3} x_{t-1} + \frac{2}{3}$; $x_0 = 2$

$x_t = a x_{t-1} + b$ where $a = 1/3$, $b = 2/3$
has the sol^n

$$x_t = a^t \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

$$\text{so } x_t = \left(\frac{1}{3}\right)^t \left(2 - \frac{2/3}{1-1/3}\right) + \frac{2/3}{1-1/3}$$

$$= \left(\frac{1}{3}\right)^t (1) + 1 = 1 + \left(\frac{1}{3}\right)^t \quad \forall t = 1, 2, \ldots$$

(c) $2x_t + 3x_{t-1} + 2 = 0$ $x_0 = -1$

$$x_t = \frac{-3}{2} x_{t-1} - \frac{2}{2}$$

$$x_t = \frac{-3}{2} x_{t-1} - 1 \quad ; \quad x_0 = -1$$

$x_t = a x_{t-1} + b$ where $a = -3/2$, $b = -1$

$$x_t = \left(\frac{-3}{2}\right)^t \left(-1 - \frac{-1}{1+3/2}\right) + \frac{-1}{1+3/2}$$

$$= \left(\frac{-3}{2}\right)^t (-3/5) - 2/5$$

Q3 $x_t = x_{t-1} + 4$; $x_0 = 2$

$x_t = a x_{t-1} + b$ where $a = 1$, $b = 4$

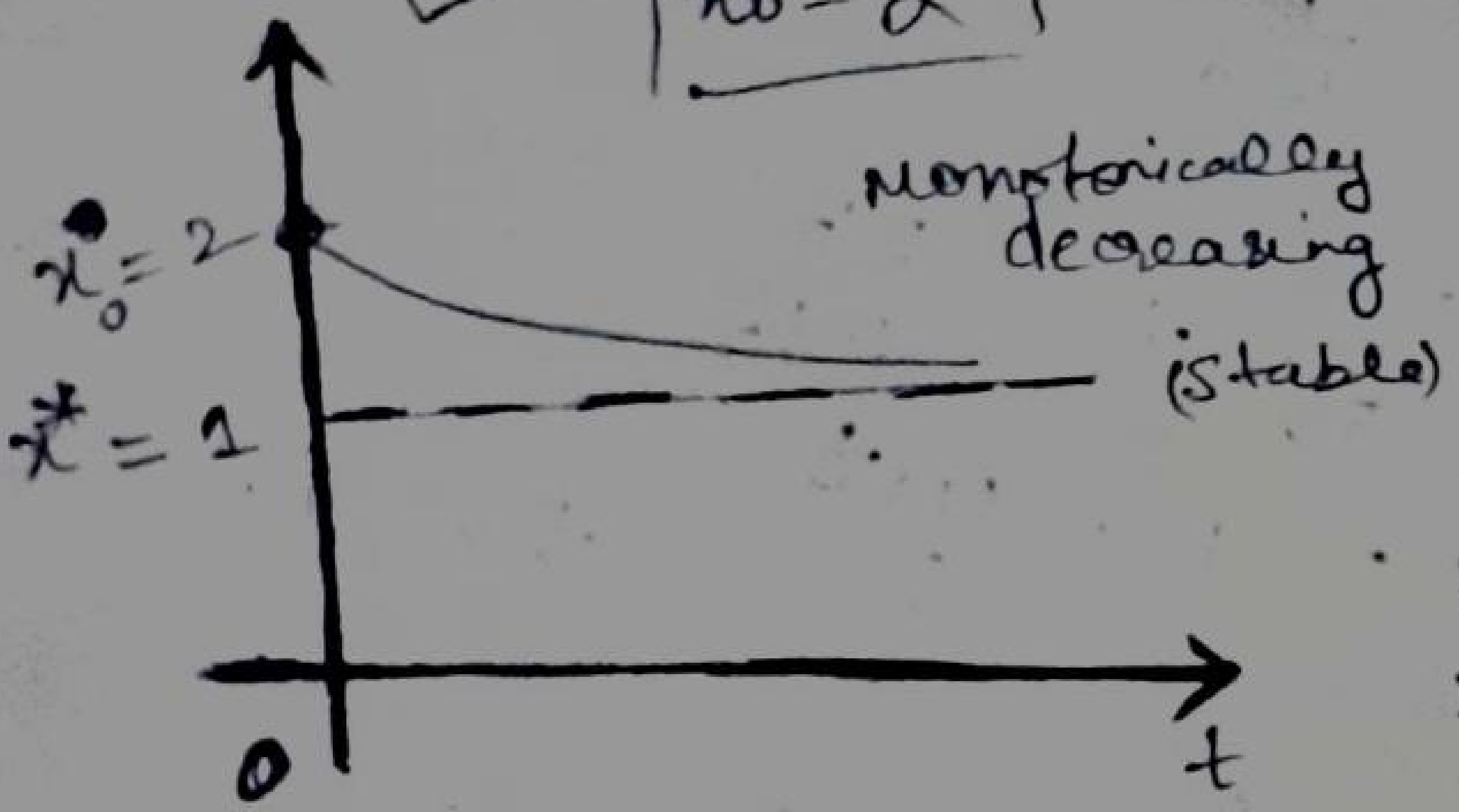
$\Updownarrow$ soln

$x_t = x_0 + t b \quad \forall t$

soln $x_t = 2 + 4t \quad \forall t$

STABILITY ANALYSIS

Q1 $\boxed{x_0 = 2}$



$\boxed{x_t = \frac{1}{3} x_{t-1} + \frac{2}{3}}$

$x^* = \frac{2/3}{1 - 1/3} = 1$

$|<1$ stable

$0 < a = \frac{1}{3} < 1$

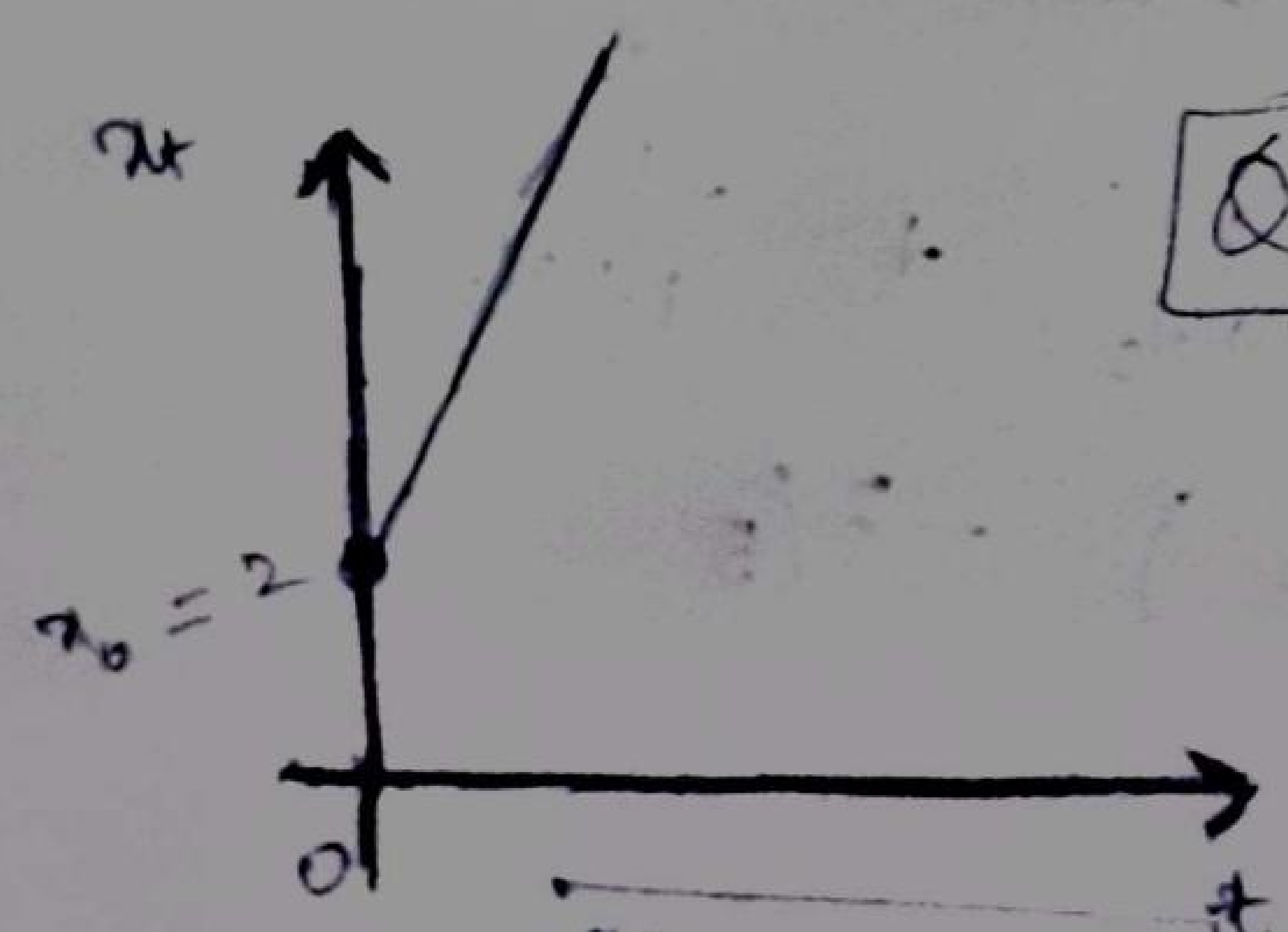
Q2 $\boxed{x_0 = -1}$

$\boxed{x_t = \frac{-3}{2} x_{t-1} - 1}$

$x^* = \frac{-1}{1 + 3/2} = -2/5$

$a = -3/2 < -1$

Q3



$\boxed{x_t = 2 + 4t}$ $x_0 = 2$

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Difference equation [Equilibrium and stability analysis]

$$x_t = a x_{t-1} + b$$

$$\Updownarrow$$

Sol^n $\quad x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \left(\frac{b}{1-a}\right) \quad (a \neq 1)$ — ①

Equilibrium (stationary) state : $x = x^*$ is such that $\forall t$ $x = x^*$

or $\quad x^* = a x^* + b$

$$\boxed{x^* = \frac{b}{1-a}} \quad \text{—— a)}$$

[once reached, nothing will change state of rest]

so for some $s \geq 0$ if $x_s = \frac{b}{1-a}$

$$x_{s+1} = a\left(\frac{b}{1-a}\right) + b$$
$$= \frac{b}{1-a}$$

Also $x_{s+2} = \frac{b}{1-a}$ & so on.

So if $x_s = \frac{b}{1-a}$ at some time s, then x_t will remain at this constant level for each $t \geq s$.

$$x_t - x^* = a x_{t-1} + b - x^*$$
$$= a x_{t-1} + b - \left(\frac{b}{1-a}\right)$$
$$= a x_{t-1} - \frac{ab}{1-a}$$

$$\therefore \boxed{x_t - x^* = a(x_{t-1} - x^*)}$$

$$\Updownarrow$$

$$\boxed{x_t - x^* = a^t(x_0 - x^*)}$$

② same as ① using a) in ①

from ② $x_t - x^*$ captures the deviation of x_t from its equilibrium state

and $= a(x_{t-1} - x^*)$

$= a^t (x_0 - x^*)$

Deviations $(x_t - x^*)$ grows (or shrinks) at the constant proportional rate $a-1$.

$$\frac{(x_t - x^*) - (x_{t-1} - x^*)}{(x_{t-1} - x^*)} = \frac{a(x_{t-1} - x^*) - (x_{t-1} - x^*)}{(x_{t-1} - x^*)}$$

$$= a - 1.$$

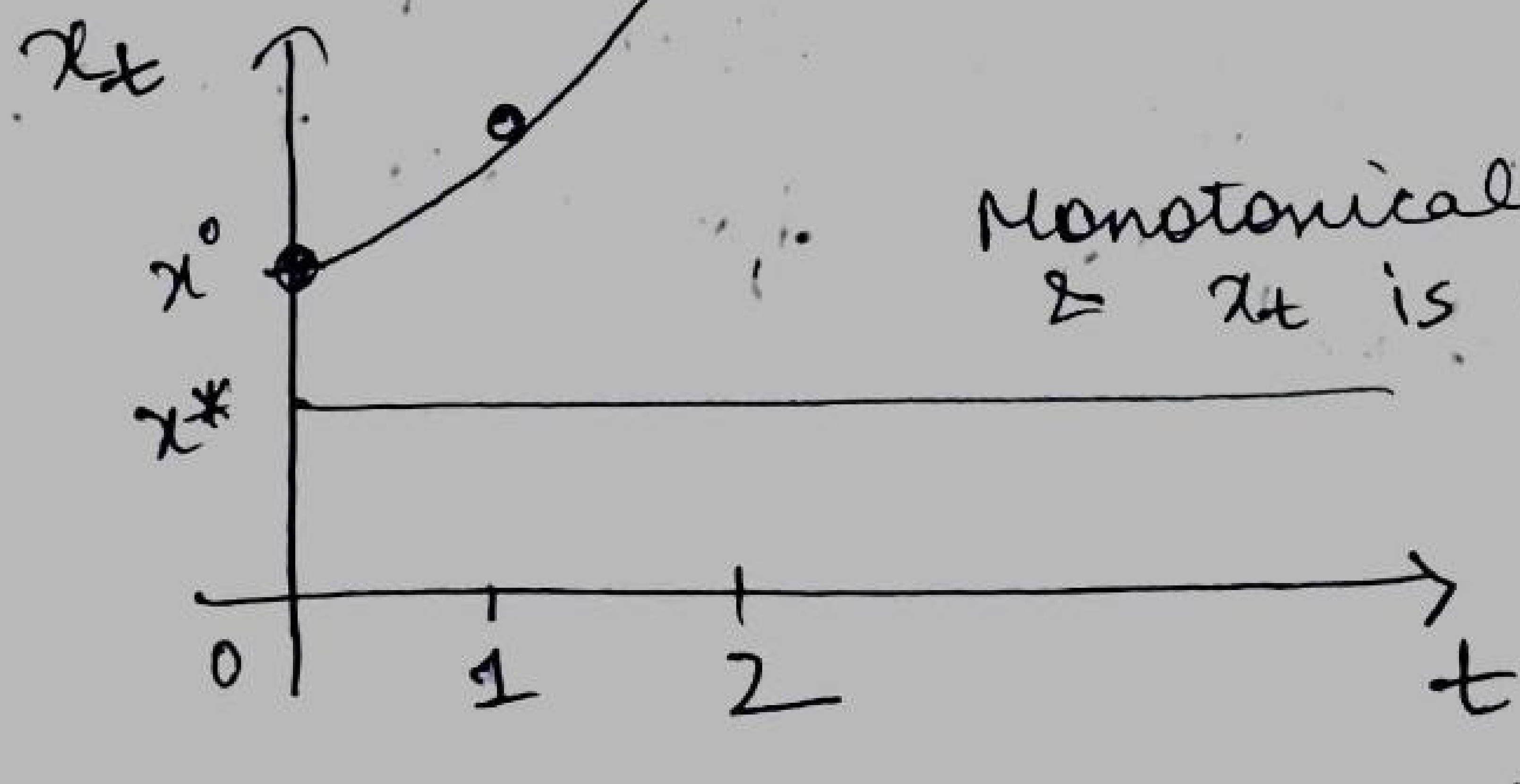
STABILITY ANALYSIS

sol^n $x_t - x^* = a^t (x_0 - x^*)$ $a \neq 1$

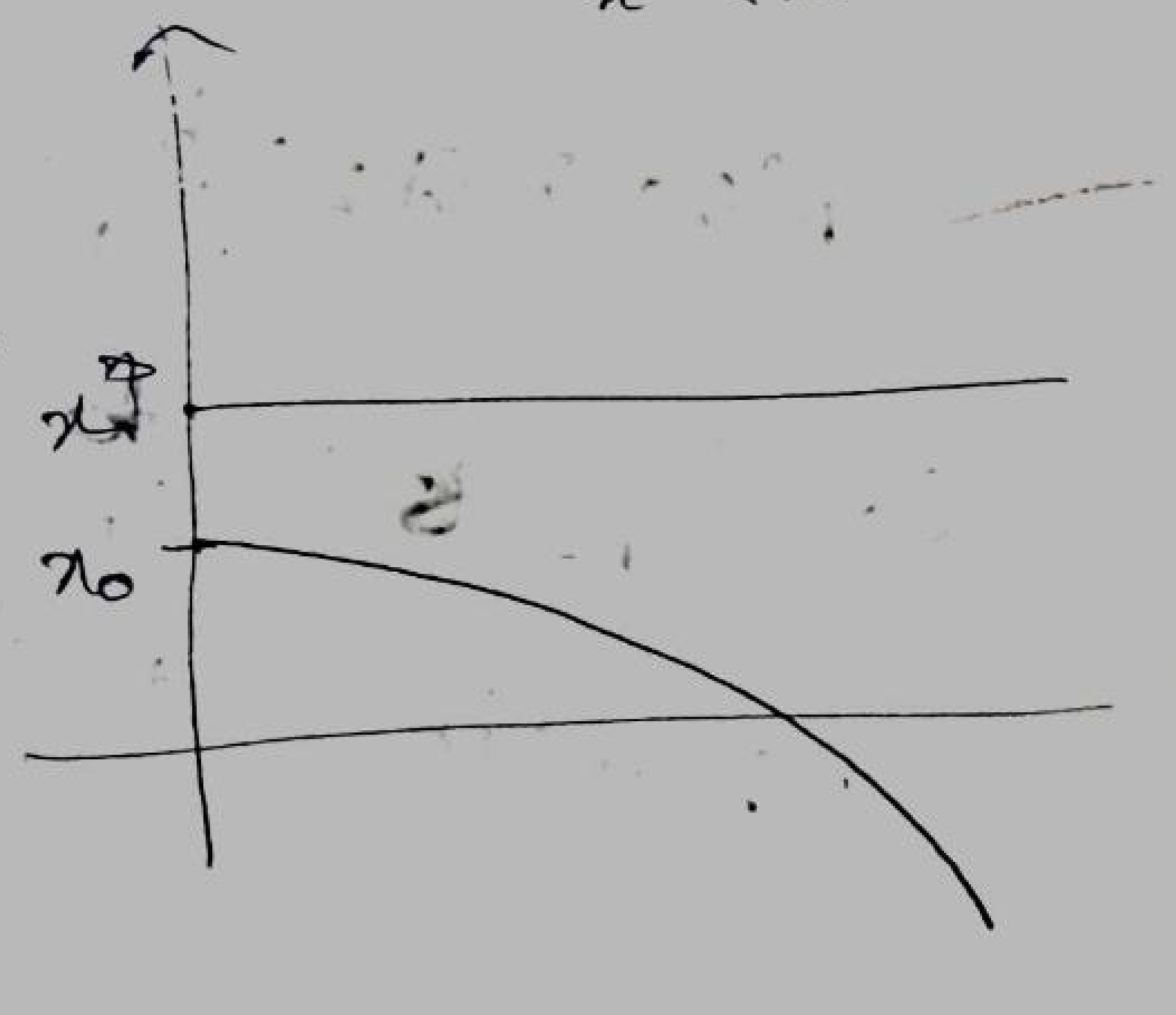
of $x_t = a x_{t-1} + b$ $\forall t$

$x^0 < x^*$

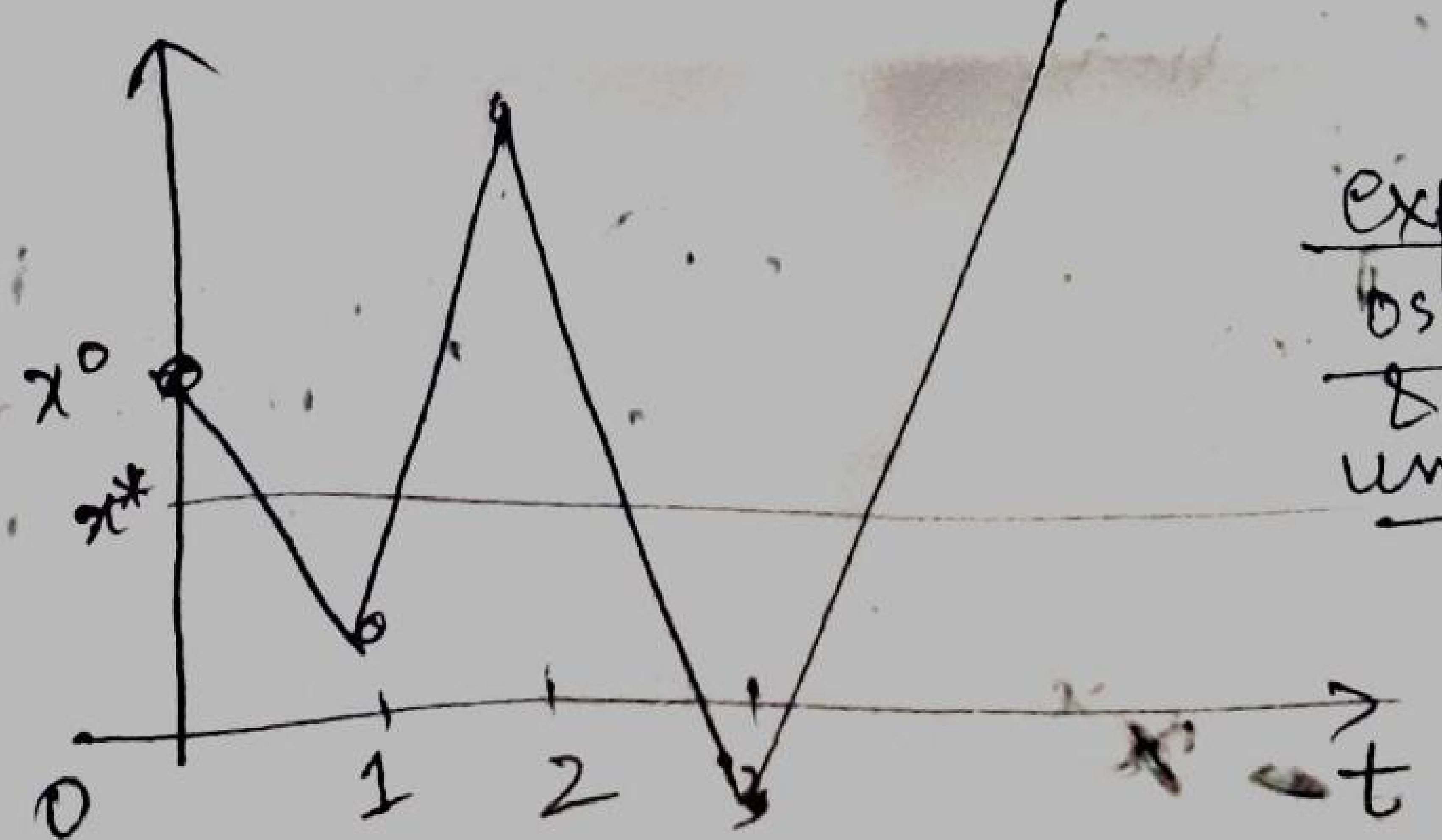
Case I $a > 1$; $x^0 > x^*$



Monotonically ↑ & x_t is unstable



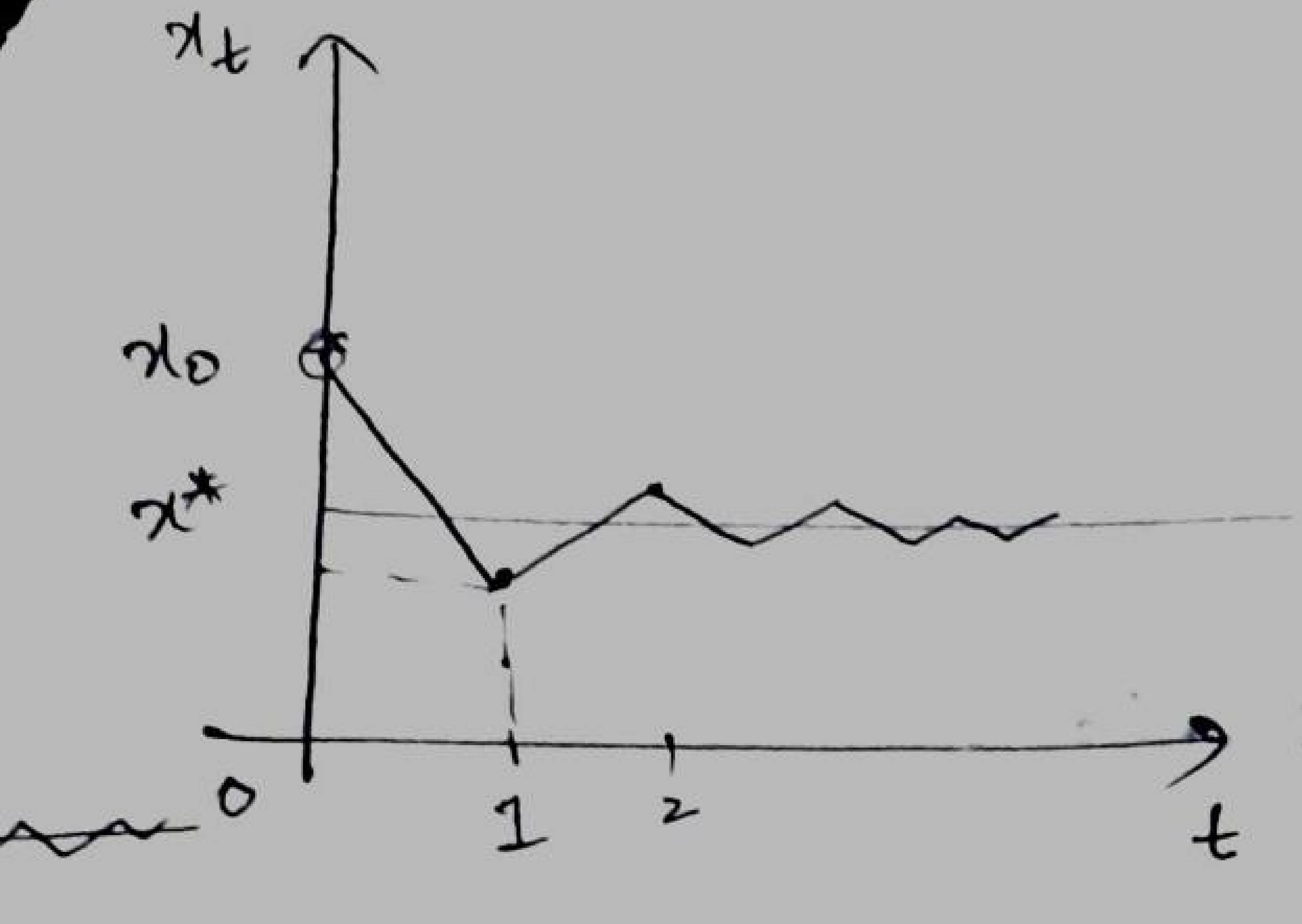
Case II $a < -1$; $x^0 > x^*$



explosive oscillating & unstable

Case III $\quad |a| < 1$

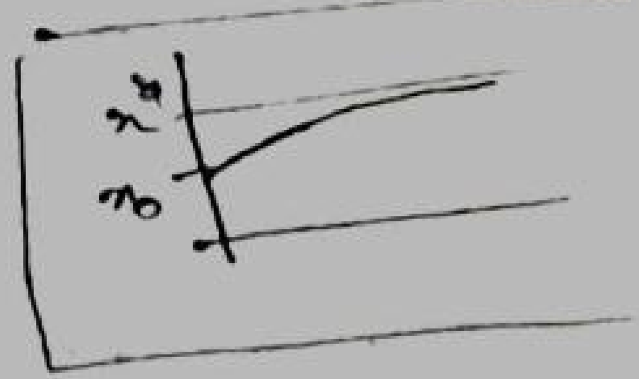
(A) $-1 < a < 0 \; ; \; x_0 > x^*$

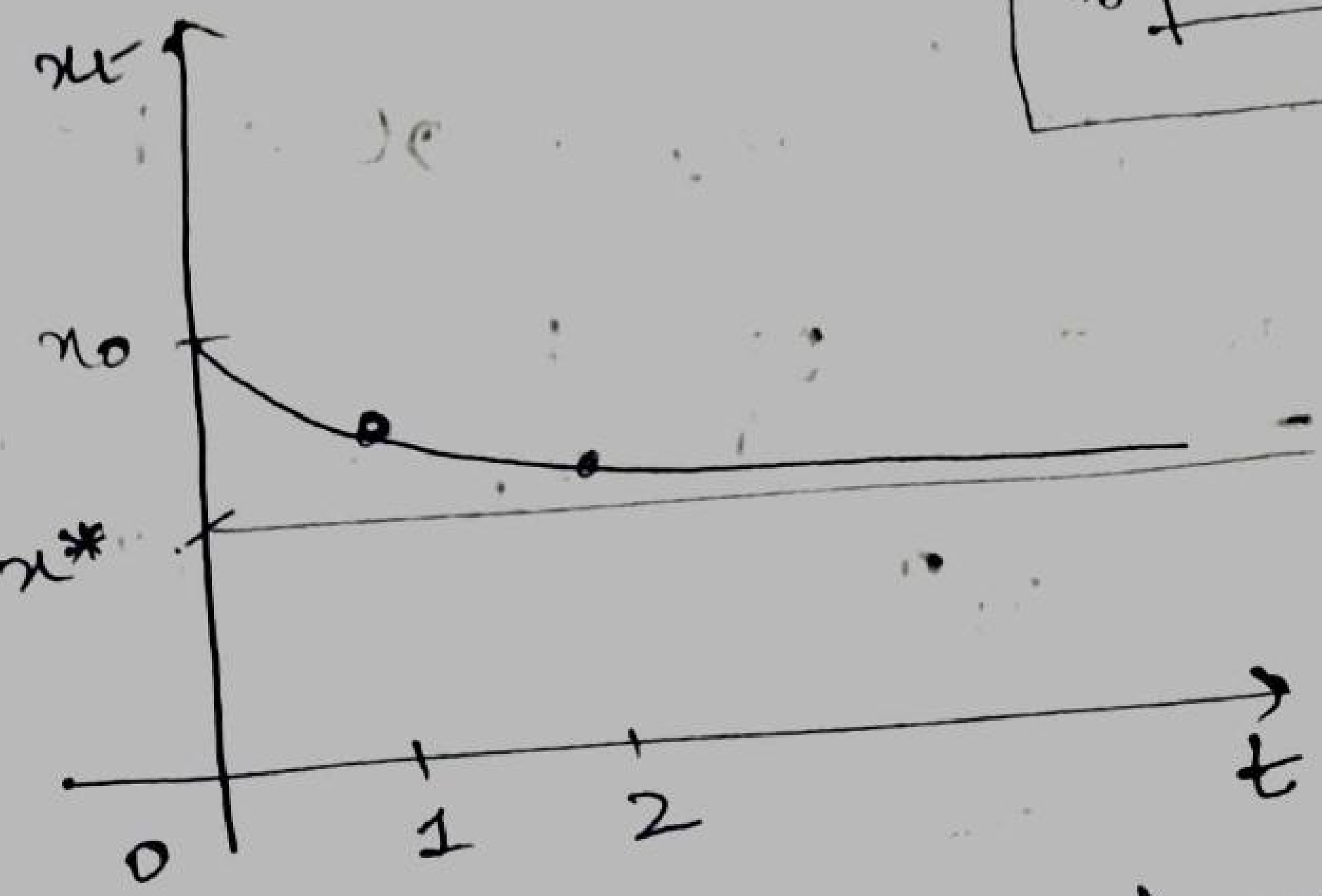


Oscillating and stable

DAMPED OSCILLATIONS

(B) $0 < a < 1 \; ; \; x_0 > x^*$

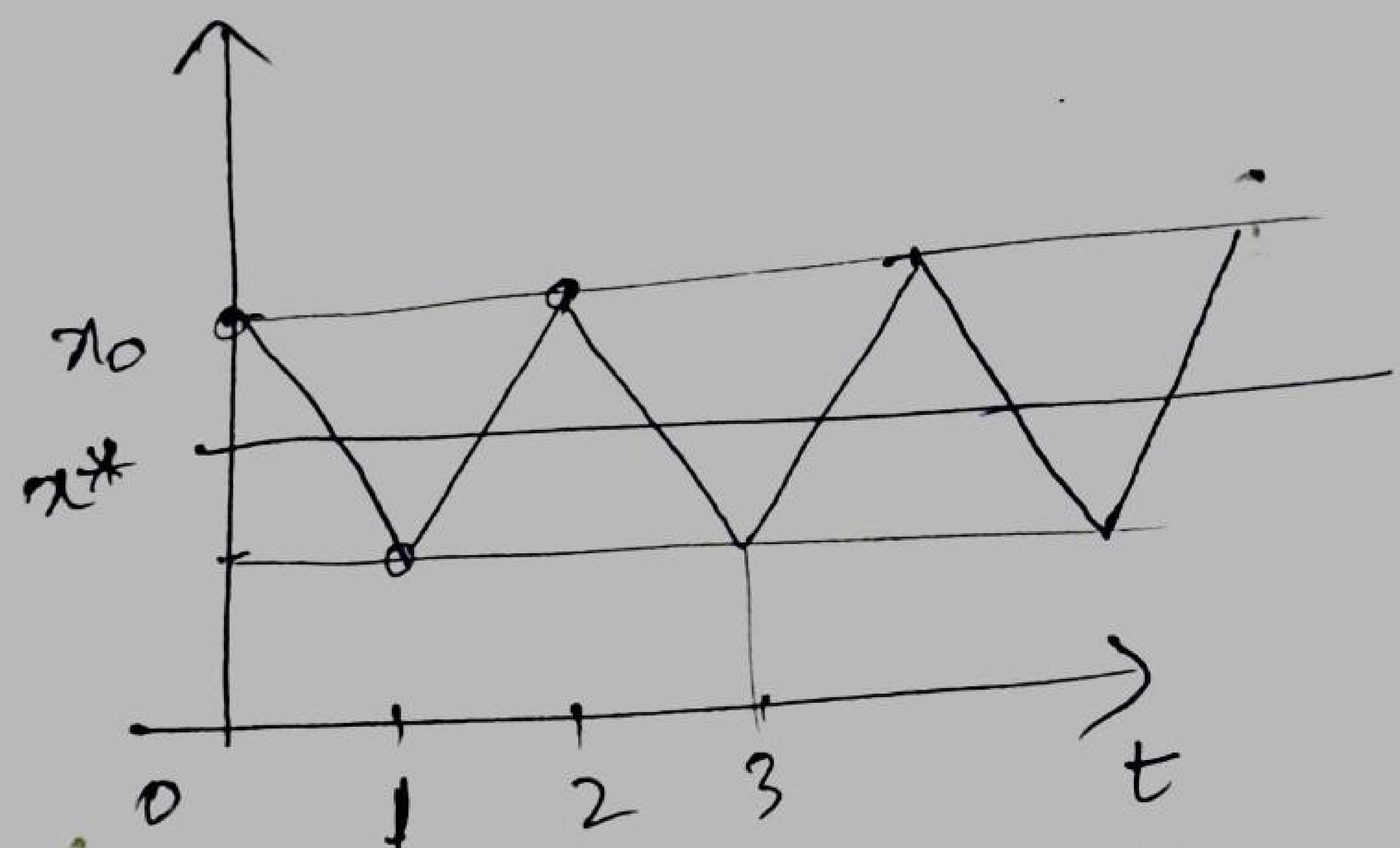




Monotonically Decreasing and stable

Case IV $\quad a = -1 \qquad x_0 > x^*$

$$x_t - x^* = (-1)^t (x_0 - x^*)$$



uniform oscillations and unstable

Suppose $a = 1$

$x_t = x_{t-1} + b$

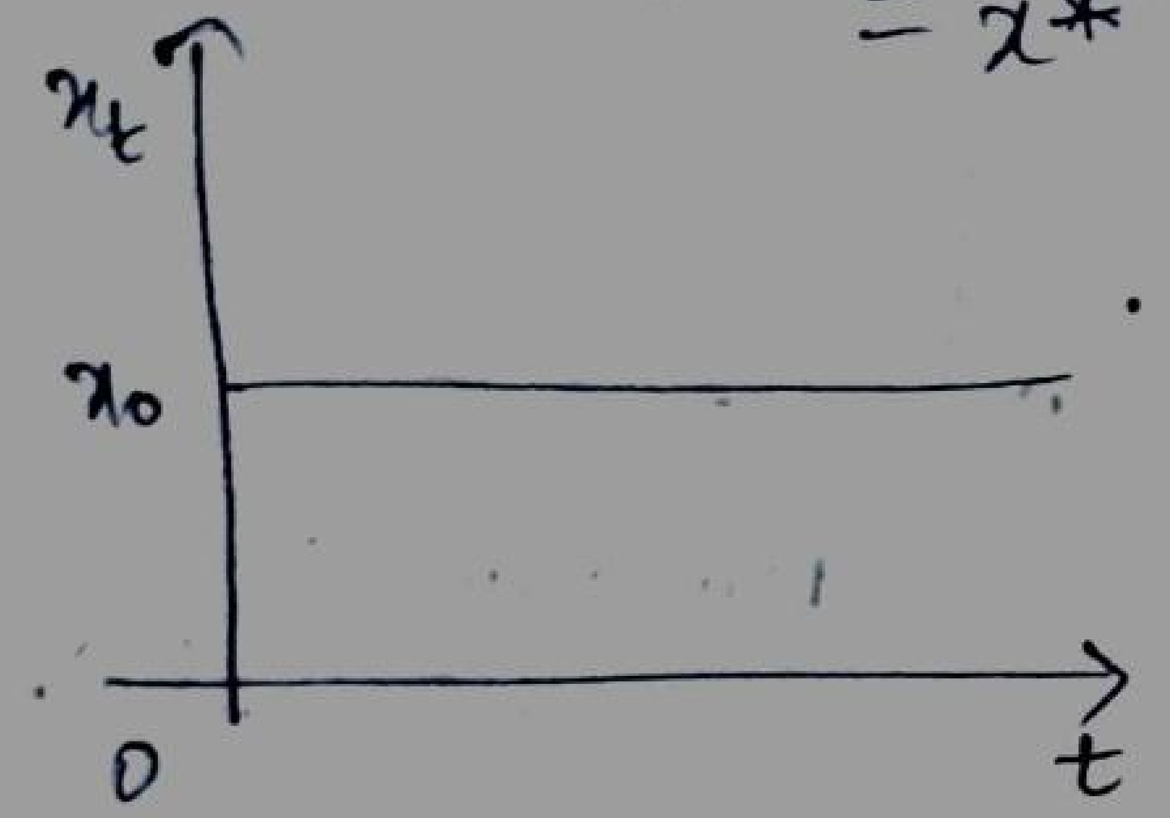
$\Updownarrow$

Sol^n $x_t = x_0 + bt$

Case I If $b = 0$

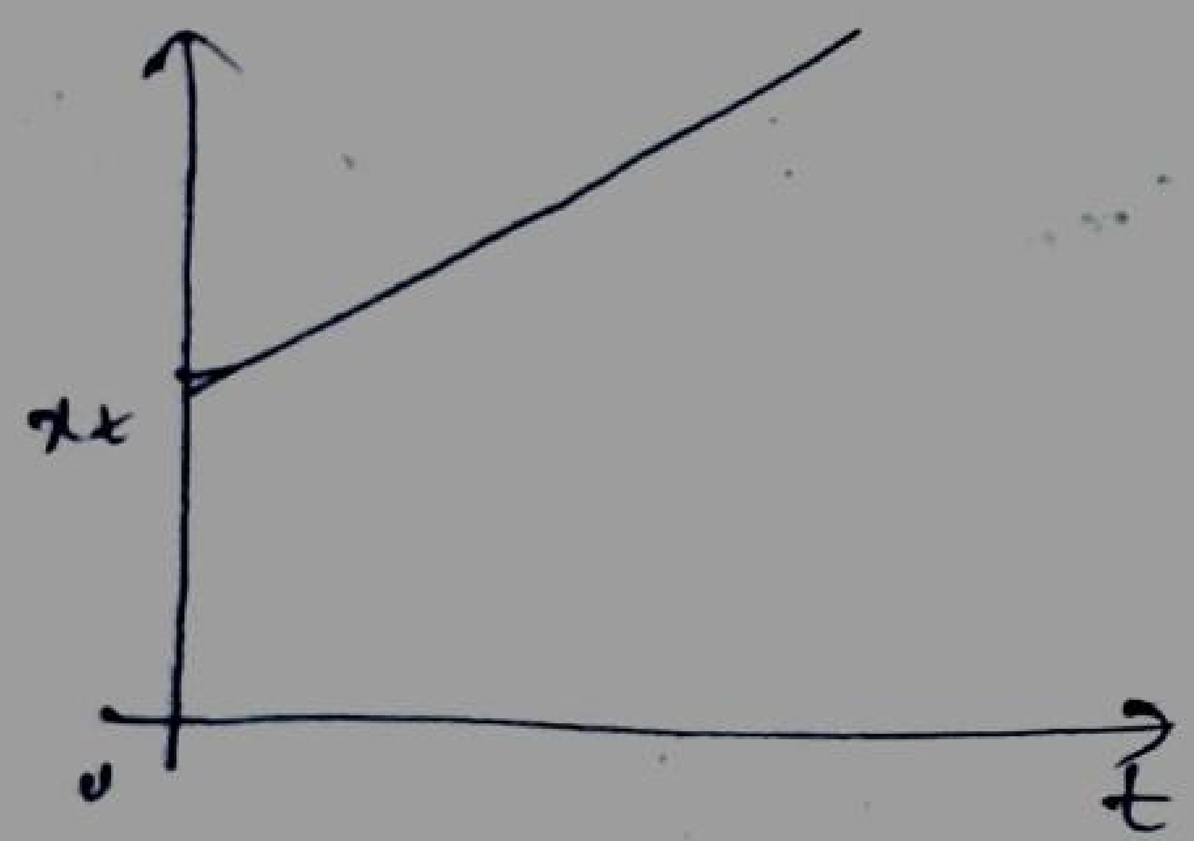
$x_t = x_0 \;\forall\; t$

$= x^*$



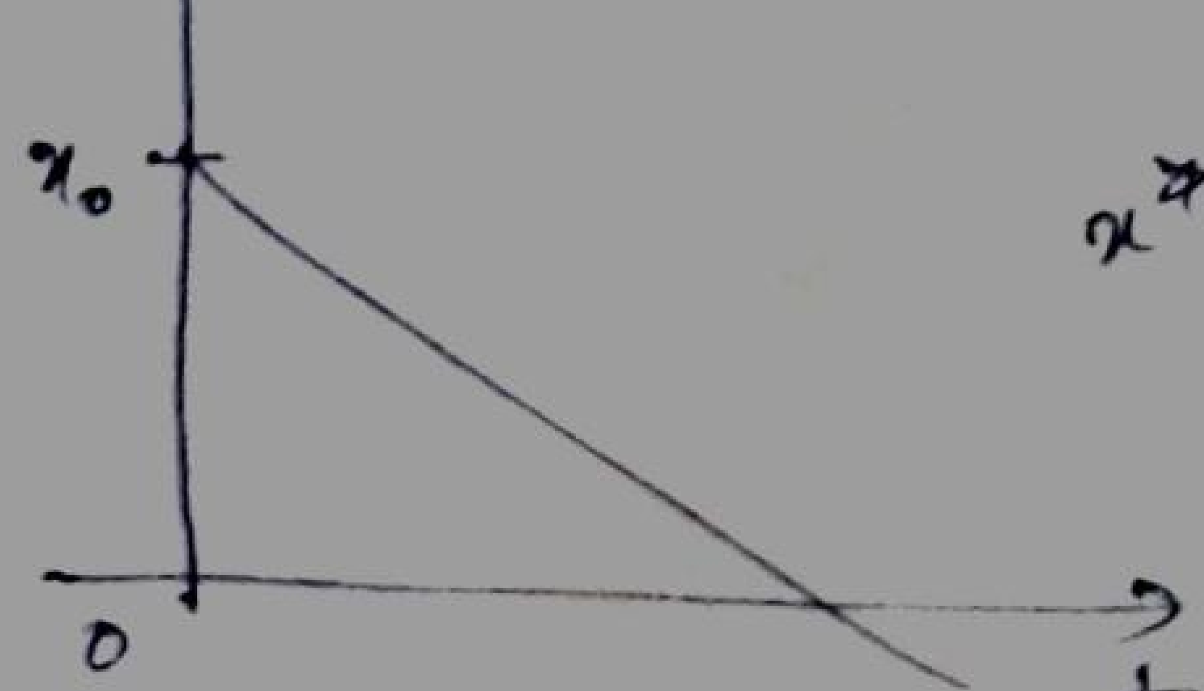
stable

Case II $b > 0$



x^* d.n.e

Case III $b < 0$



x^* d.n.e.

Imp

1. x_t will be stable & converge to eq^n if $|a| < 1$
2. x_t will be unstable if $|a| > 1$
3. x_t will oscillate if $a < 0$
4. x_t will change montonically if $a > 0$

Cobweb Model

Consider a situation in which producer's output-decision must be made one period in advance of the actual sales, such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output.

Output in period 't' is determined by P_{t-1}

or $\quad Q_{st} = S(P_{t-1})$ $\quad$ Supply function

Demand function $\quad Q_{dt} = D(P_t)$.

In particular we assume;

$$Q_{dt} = \alpha - \beta P_t \quad (\alpha, \beta > 0) \quad (1)$$

$$Q_{st} = -\gamma + \delta P_{t-1} \quad (\gamma, \delta > 0) \quad (2)$$

Equilibrium condition is given by $\quad Q_{dt} = Q_{st}$ (3)

Substitute (1) n (2) in (3), we get

$$\alpha - \beta P_t = -\gamma + \delta P_{t-1}$$

$$\boxed{P_t = -\frac{\delta}{\beta} P_{t-1} + \frac{(\gamma + \alpha)}{\beta}}$$

given $P_0 > \overline{P}$

Sol^n $\quad P_t = \left(-\frac{\delta}{\beta}\right)^t \left(P_0 - \frac{\alpha+\gamma}{\beta+\delta}\right) + \left(\frac{\alpha+\gamma}{\beta+\delta}\right) \quad \forall t$

$$\boxed{P^* = \frac{\alpha+\gamma}{\beta+\delta}} = \overline{P}$$

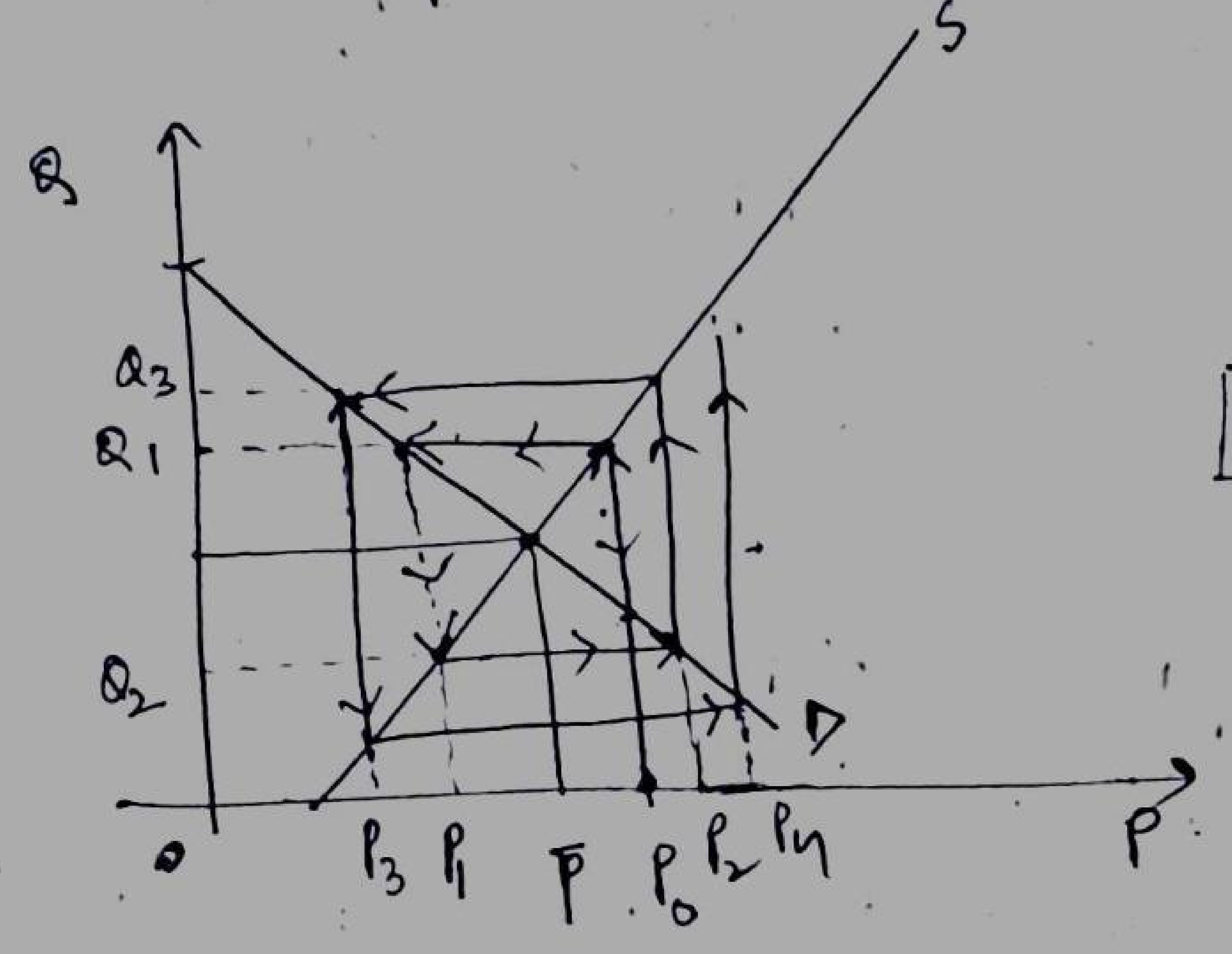
$$P_t = \left(-\frac{\delta}{\beta}\right)^t (P_0 - \overline{P}) + \overline{P} \quad ; \quad P_0 > \overline{P}$$

$a = -\frac{\delta}{\beta} < 0 \Rightarrow P_t$ will oscillate

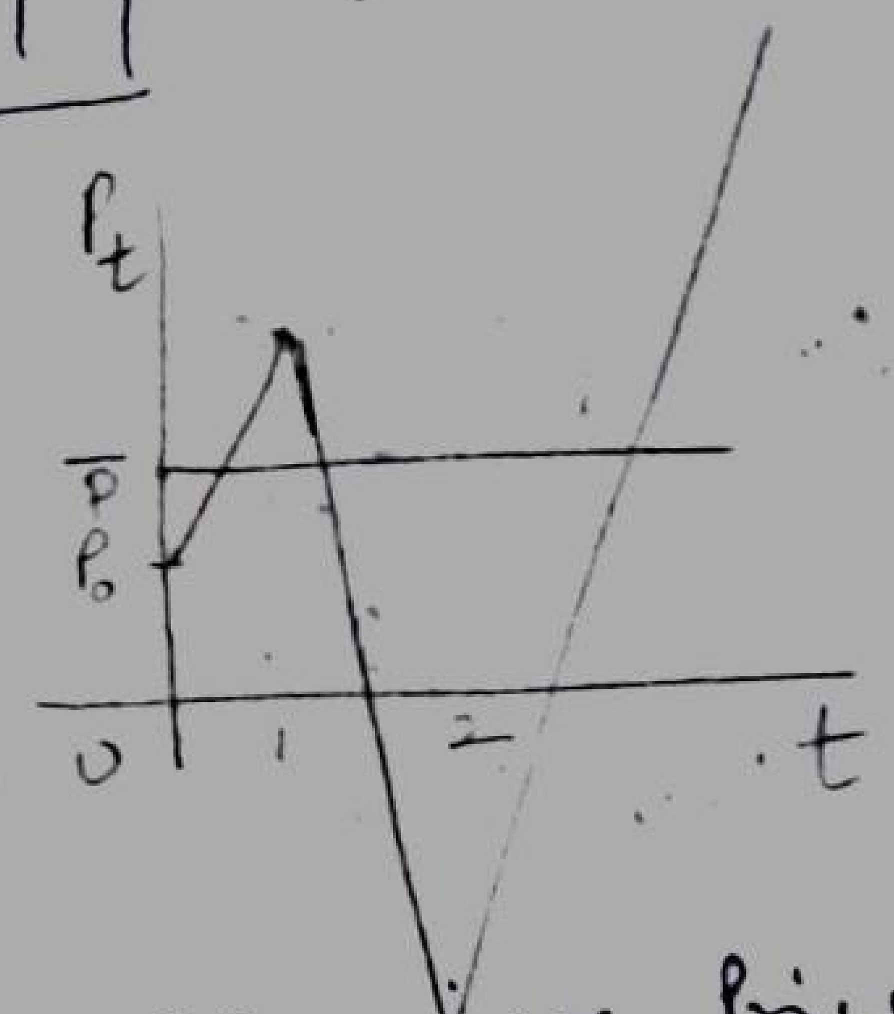
<u>Case I</u> $\delta > \beta$ ie $a = -\frac{\delta}{\beta} < -1$

(Explosive Oscillations)

<u>Supply curve is steeper than demand curve</u>



$\boxed{a < -1}$



P_0 - initial price

given P_0 — In pd 1 $Q_s = Q_1$ which gives price

For Eq^n to be estab. in pd 1 $Q_d = Q_1$

in pd 1 $= P_1$

By comparing the price levels $P_0, P_1, P_2, \ldots$, we observe oscillatory pattern of change and a tendency for price to widen its deviation from $\overline{P}$. With cobweb being spun from inside out, time path is divergent & oscillation explosive

$a = -\frac{\delta}{\beta} < 0$ OSCILLATIONS

$a < -1$ — Explosive & unstable

$-\frac{\delta}{\beta} < -1$

$\boxed{\delta > \beta}$

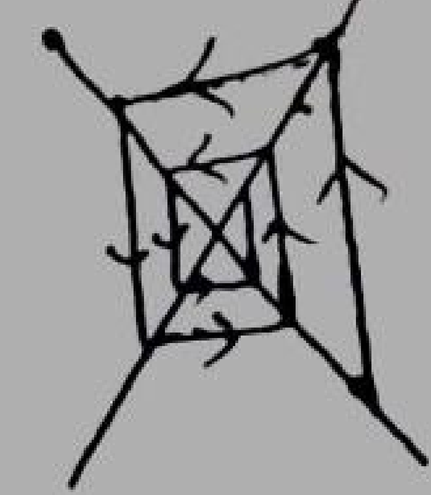
$-1 < a < 0$ — Damped oscillation stable

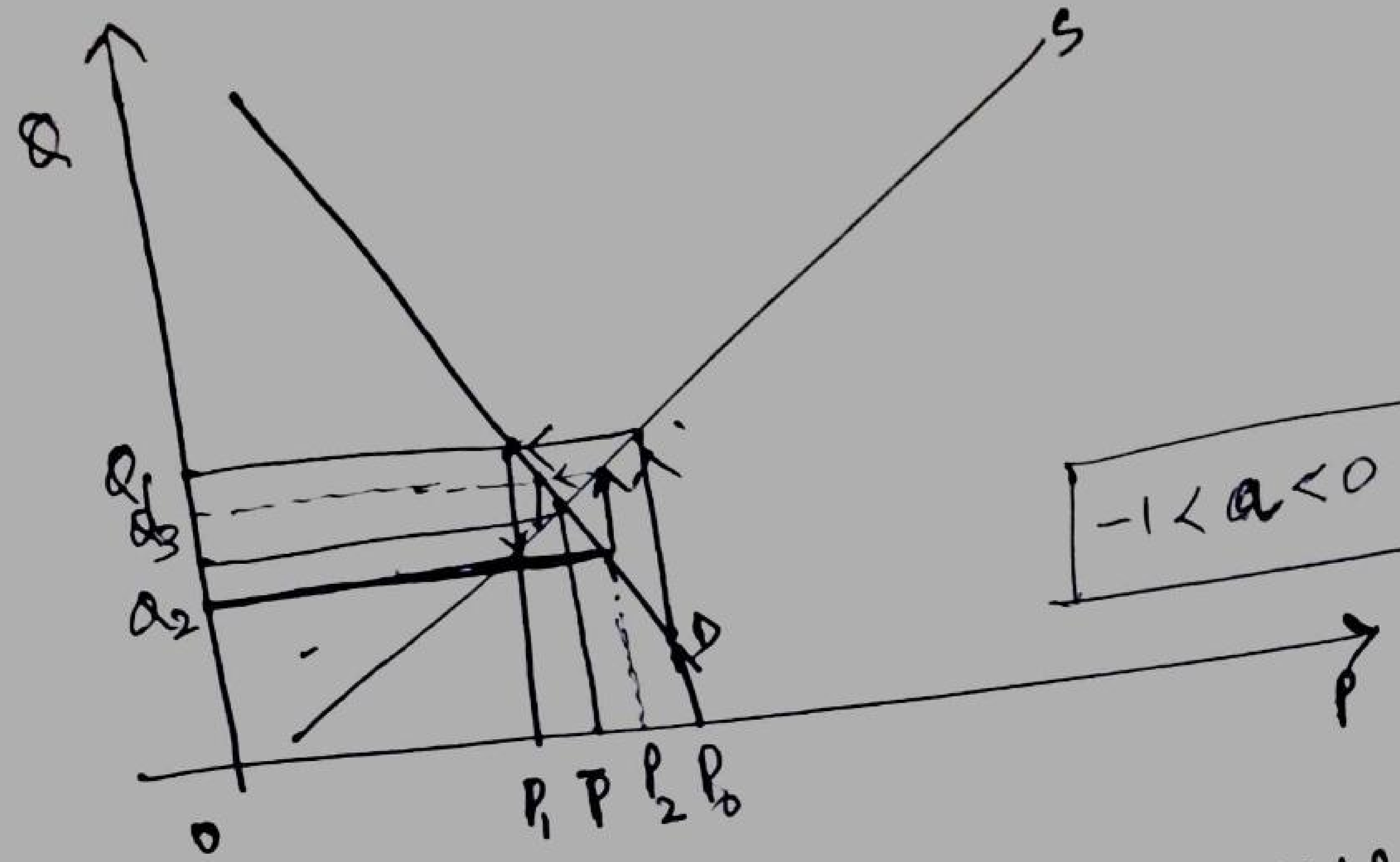
$\boxed{\delta < \beta}$

$\boxed{\delta = \beta}$ $a = -1$ ⟶ uniform unstable

$\delta < \beta \qquad 0 > a = -\dfrac{\delta}{\beta} > -1$

Zoom In

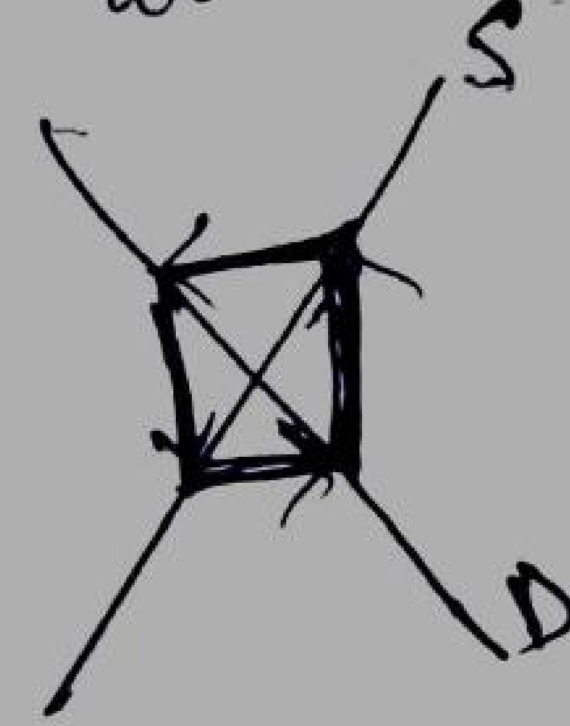




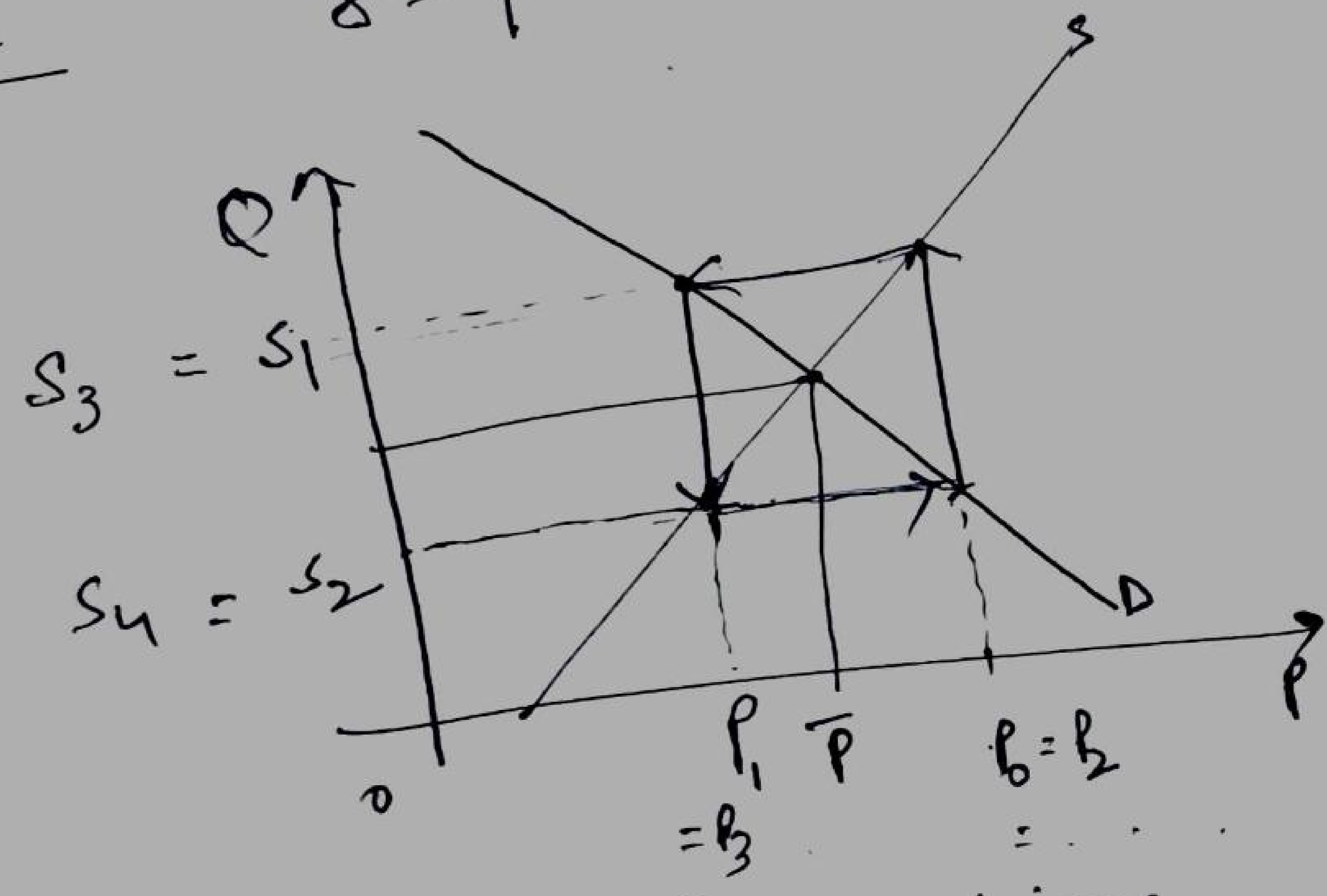
$\boxed{-1 < a < 0}$

Path is oscillatory & convergent
DAMPED

Zoom In



Case III $\qquad \delta = \beta$



$\boxed{a = -1}$

uniform oscillations